Hard problems in post-quantum cryptography

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DISTANT Training School
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* Animated version at: https://mtrimoska.com/slides/PQC_DISTANT/index.html

Post-quantum cryptography

Implemented on a classical, but resistant to attacks on a quantum computer.

- Shor's quantum algorithm: solves integer factorisation and discrete logarithms in abelian groups in polynomial time.
  - All* currently deployed public-key cryptosystems would be broken by an adversary in possession of a large quantum computer.
  - All public-key cryptosystems need to be replaced.

Post-quantum cryptography

Implemented on a classical, but resistant to attacks on a quantum computer.

- Shor's quant… …nial time.
  - All* cu… …a large quant…
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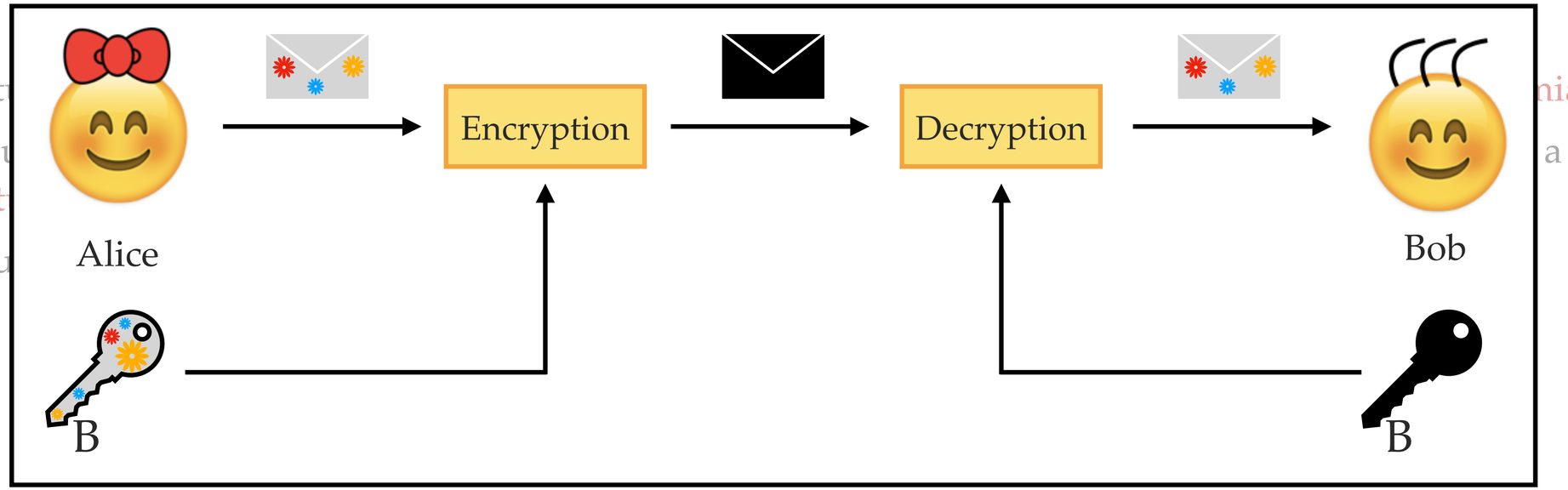

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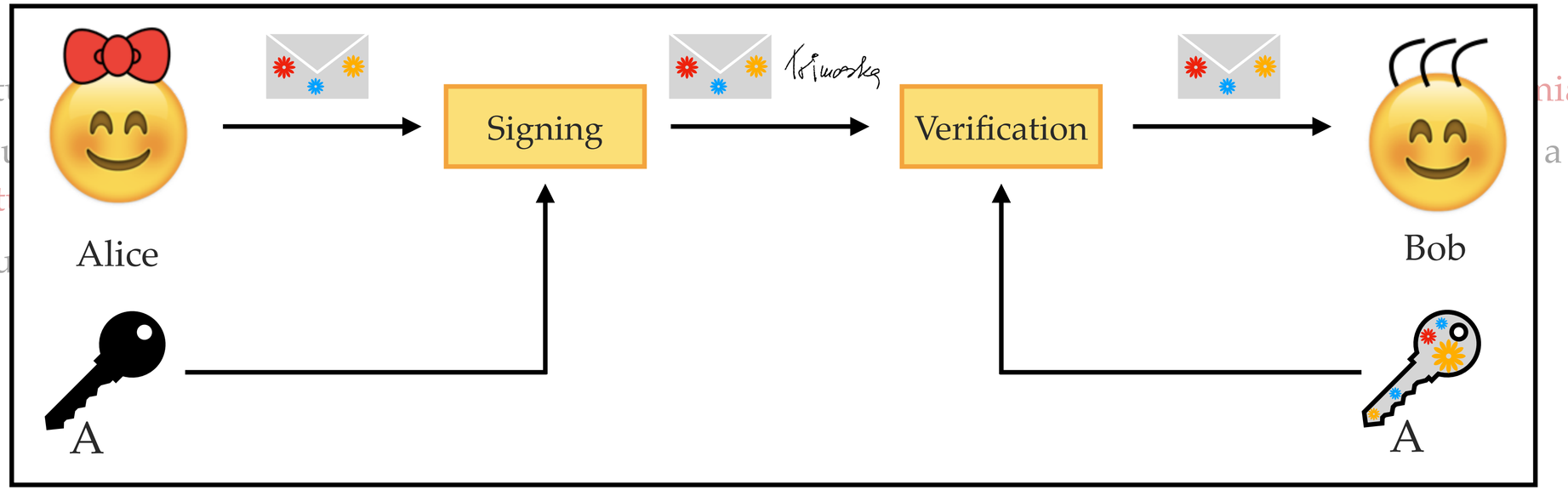


Post-quantum cryptography

Implemented on a classical, but resistant to attacks on a quantum computer.

- Shor's quantum algorithm: solves integer factorisation and discrete logarithms in abelian groups in polynomial time.
  - All* currently deployed public-key cryptosystems would be broken by an adversary in possession of a large quantum computer.
  - All public-key cryptosystems need to be replaced.
  - If the public-key cryptography component is broken, the entire infrastructure is broken because the handshake is compromised.
- Grover's quantum algorithm: quadratic speedup of exhaustive search.
  - Impact on symmetric cryptography (as a rule of thumb): double the key sizes.

Computationally hard problems

Discrete log problem

Travelling salesman problem

MQ (multivariate quadratic) problem

Isomorphism of polynomials problem

Boolean satisfiability problem

Code equivalence problem

Integer factorisation problem

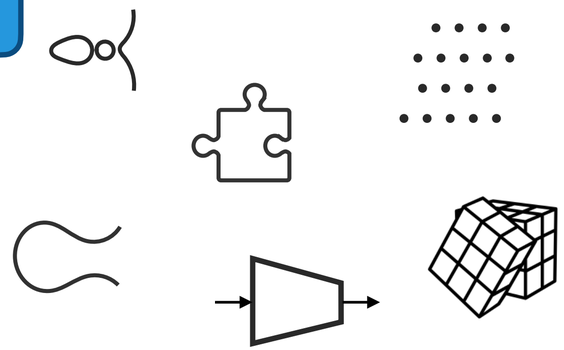
Graph colouring problem

Isogeny path problem

Syndrome decoding problem

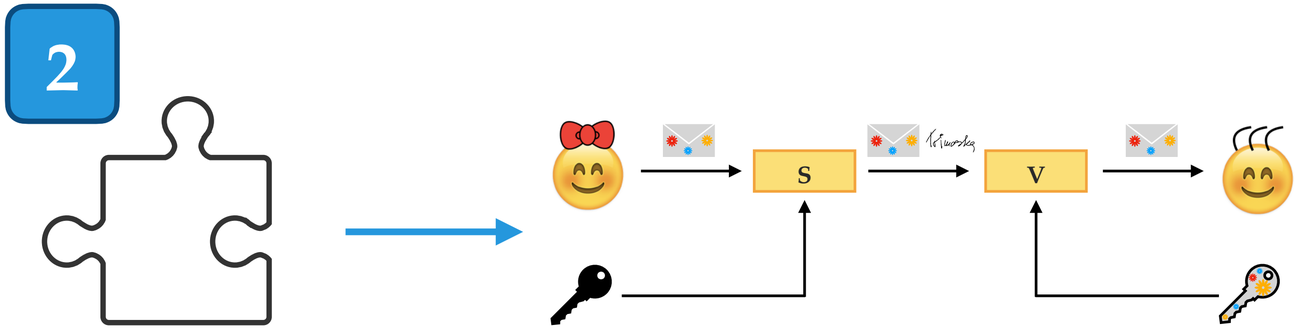
In this talk

1



The different flavours of PQC

2

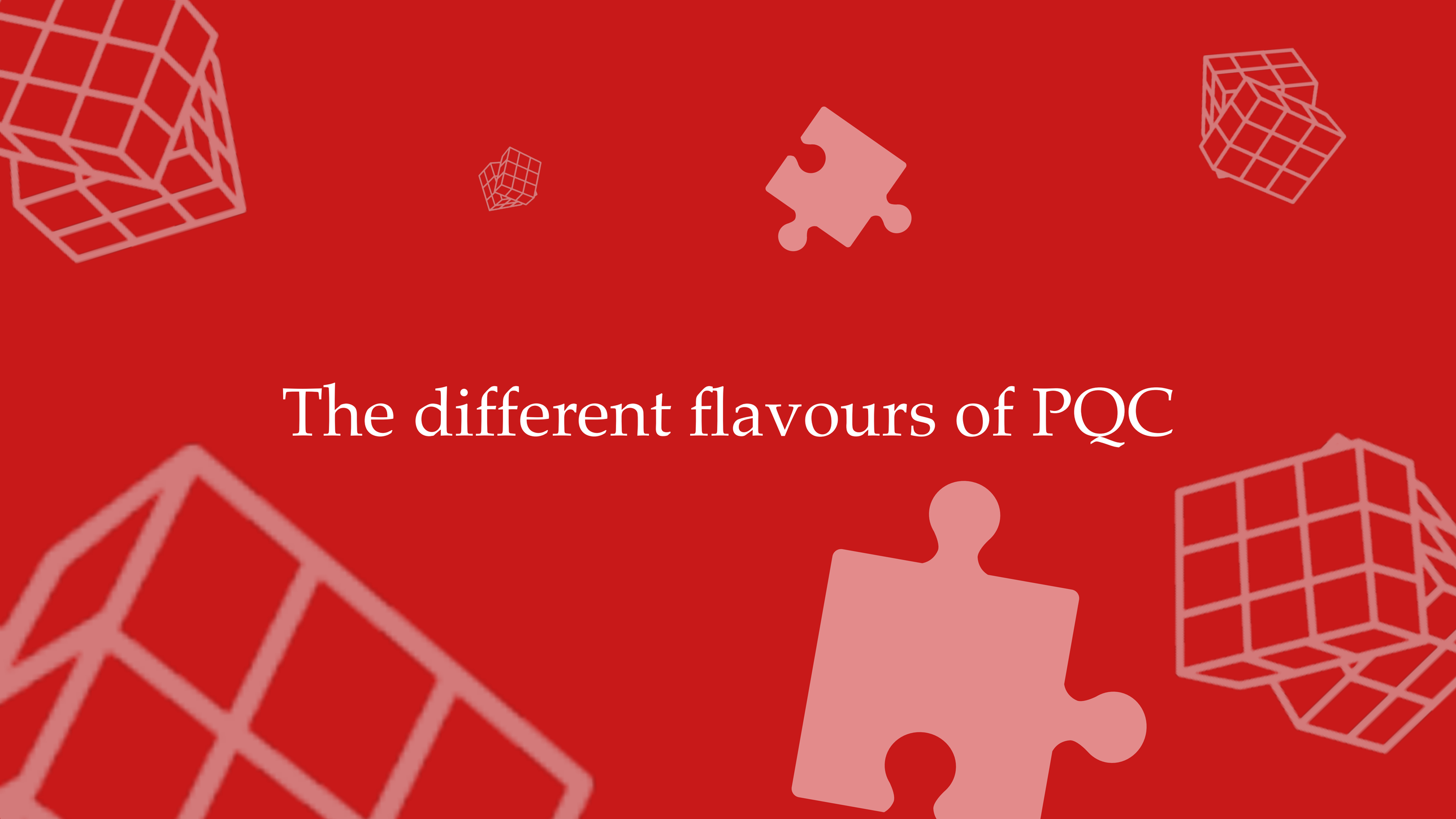


Cryptographic design example

3

Timeline

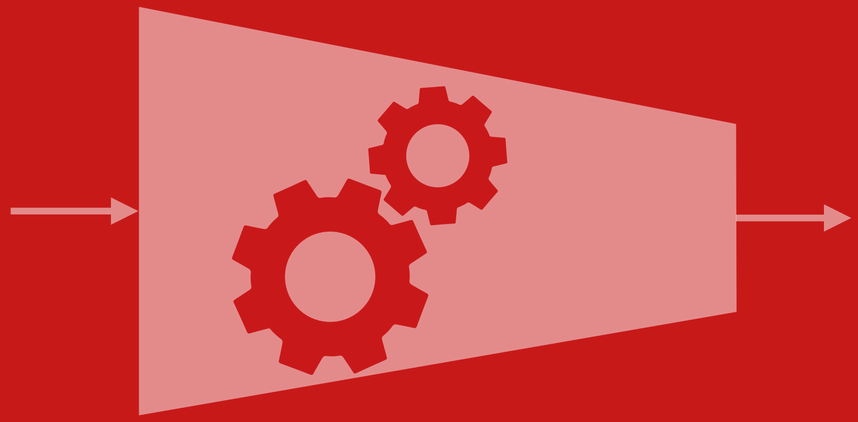
The different flavours of PQC



PQC families

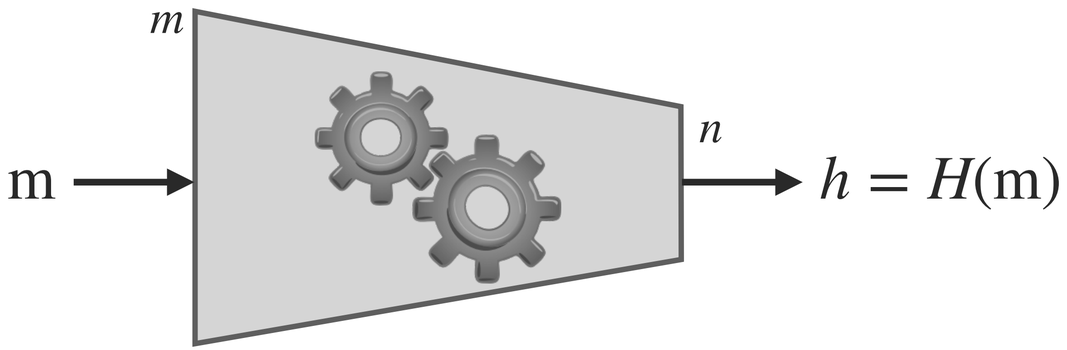
- Hash-based cryptography
- Multivariate cryptography
- Code-based cryptography
- Lattice-based cryptography
- Isogeny-based cryptography

Hash-based cryptography



Hash-based cryptography

Worst-case complexity: $\mathcal{O}(2^n)$



Hard problem: find a pre-image of h.

Used to build digital signature schemes with only one security assumption.

Multivariate cryptography

Multivariate cryptography

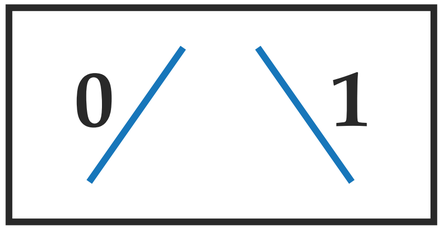
The MQ problem

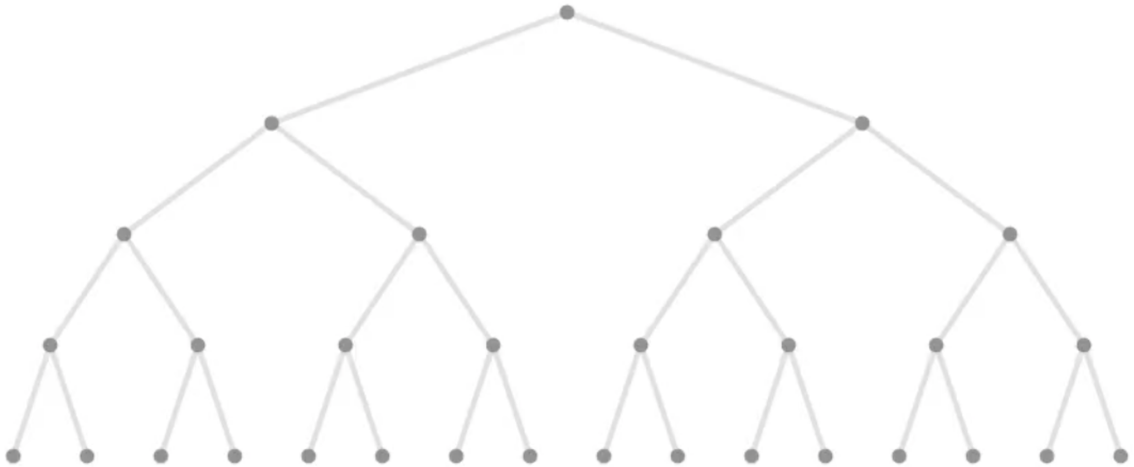
Input: m multivariate quadratic polynomials $f_1, \ldots, f_m$ of n variables over a finite field $\mathbb{F}_q$.
Question: find a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{F}_q^n$, such that $f_1(\mathbf{x}) = \ldots = f_m(\mathbf{x}) = 0$.

Example.

$$f_1 : x_1x_3 + x_2x_4 + x_1 + x_3 + x_4 = 0$$
$$f_2 : x_2x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_4 = 0$$
$$f_3 : x_2x_4 + x_3x_4 + x_1 + x_3 + 1 = 0$$
$$f_4 : x_1x_2 + x_1x_3 + x_2x_3 + x_3 + x_4 + 1 = 0$$
$$f_5 : x_1x_2 + x_2x_3 + x_1x_4 + x_3 = 0$$
$$f_6 : x_1x_3 + x_1x_4 + x_3x_4 + x_1 + x_2 + x_3 + x_4 = 0$$

Multivariate cryptography





$$x_1 \cdot x_2 + x_1 \cdot x_3 + x_3 \cdot x_4 + x_3 = 0$$

$$x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_2 + 1 = 0$$

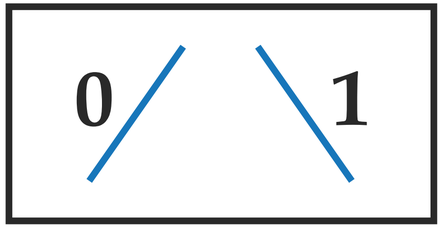
$$x_1 \cdot x_2 + x_2 \cdot x_3 + x_2 \cdot x_4 + x_1 + x_4 = 0$$

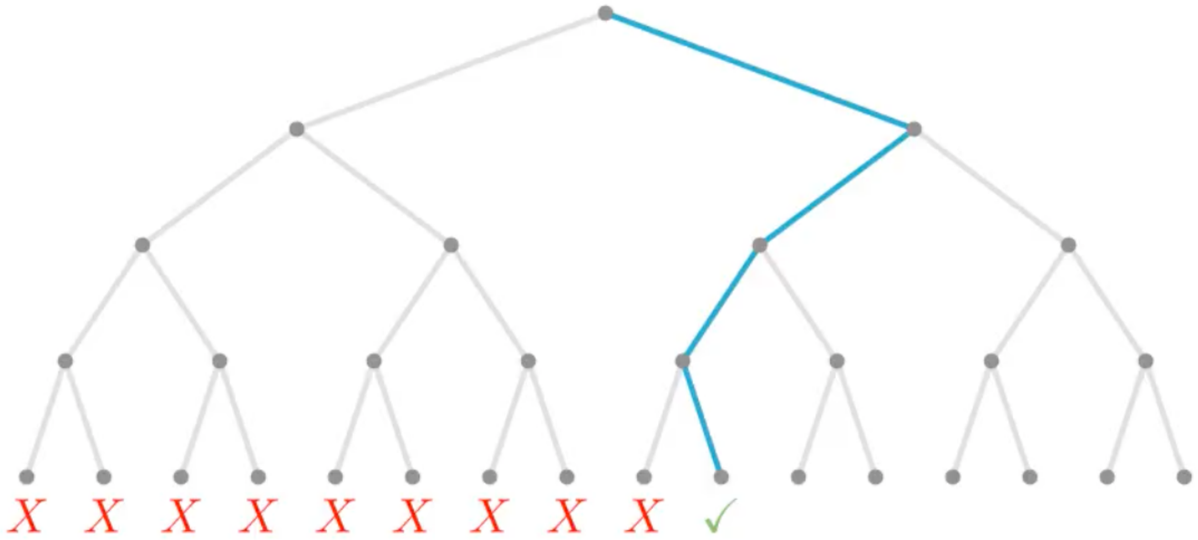
$$x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 + x_3 + x_4 = 0$$

Binary search tree

Multivariate cryptography

Worst-case complexity: $\mathcal{O}(2^n)$





Binary search tree

$$1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + 0 = 0$$

$$0 \cdot 0 + 0 \cdot 1 + 1 + 0 + 1 = 0$$

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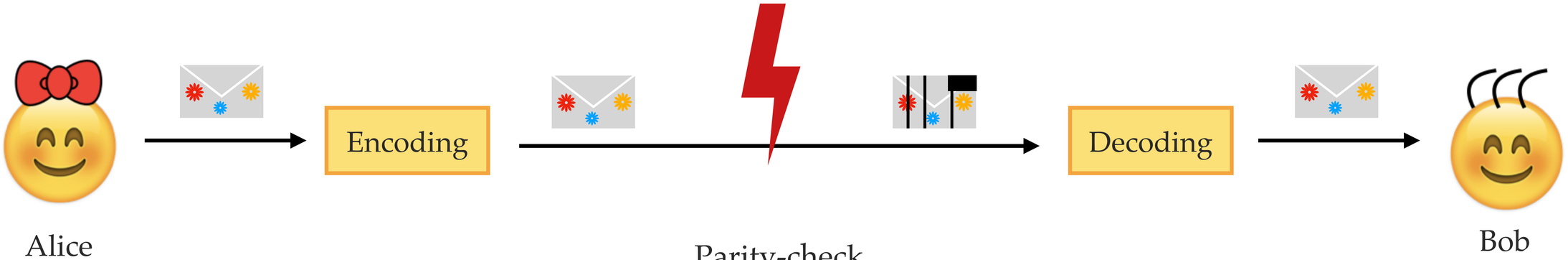
Code-based cryptography

Code-based cryptography



Parity-check matrix

Codeword: it is in the kernel of $\mathbf{H}$

$$\mathbf{Hc} = \mathbf{0}$$

Small error: Hamming weight is t

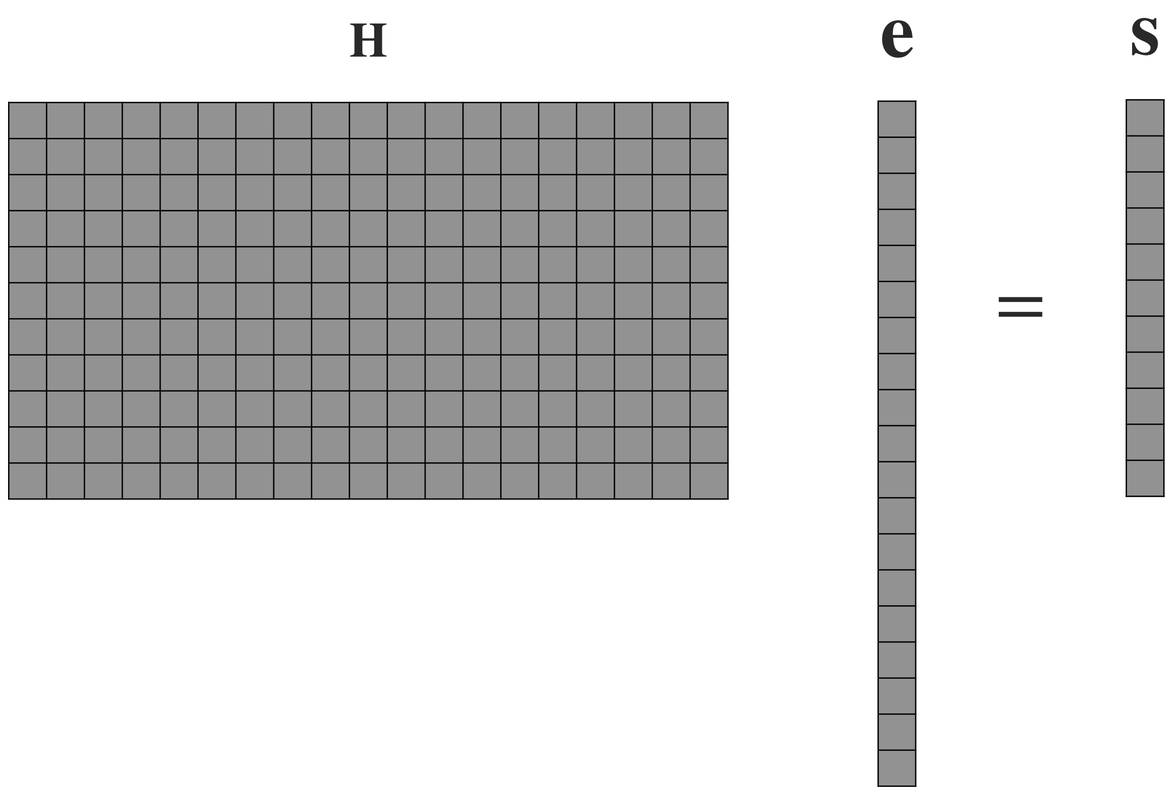
Syndrome: depends only on the error vector.

$$\mathbf{H}(\mathbf{c}+\mathbf{e}) = \mathbf{Hc} + \mathbf{He} = \mathbf{0} + \mathbf{He} = \mathbf{He} = \mathbf{s}$$

The syndrome decoding problem

Given a syndrome $\mathbf{s} = \mathbf{He}$, find $\mathbf{e}$ such that $\mathrm{wt}(\mathbf{e}) = t$.

Code-based cryptography

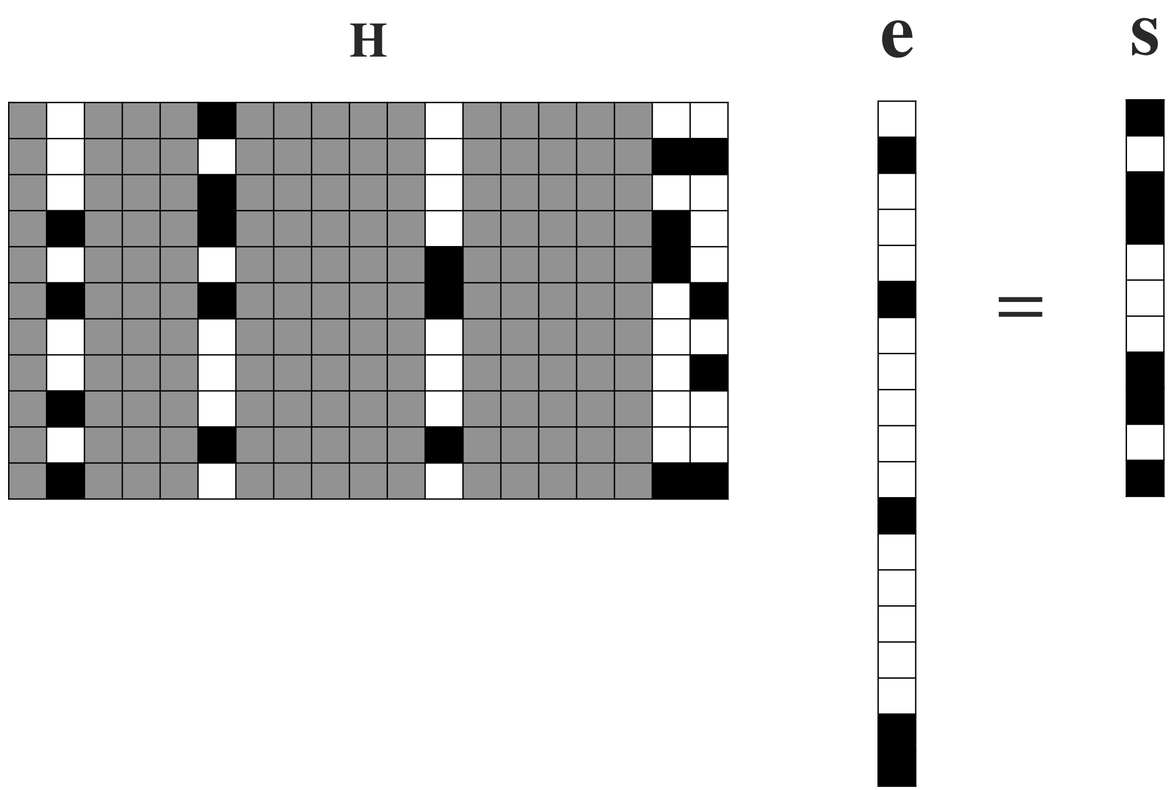


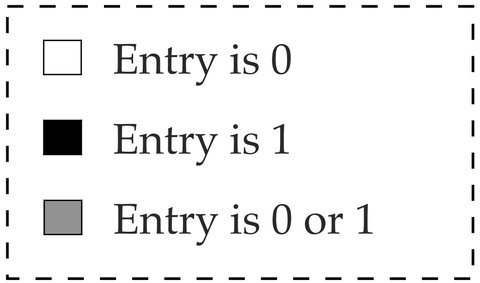
Entry is 0

Entry is 1

Entry is 0 or 1

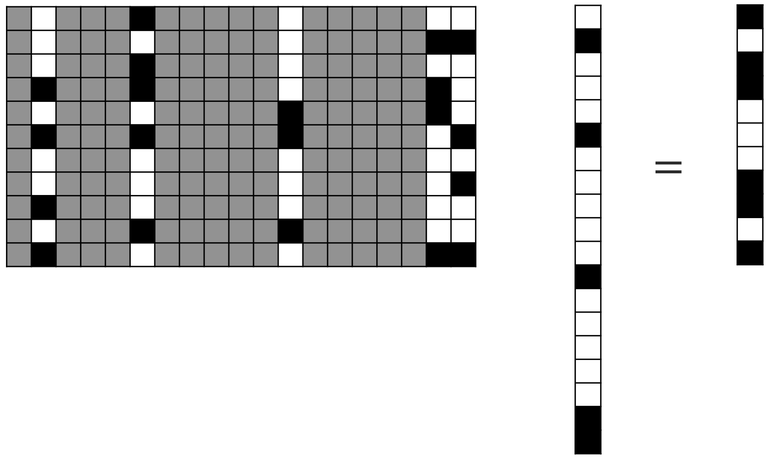
Code-based cryptography





s is equal to the sum of the columns where e_i is nonzero.

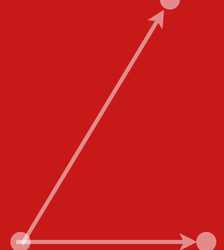
Code-based cryptography



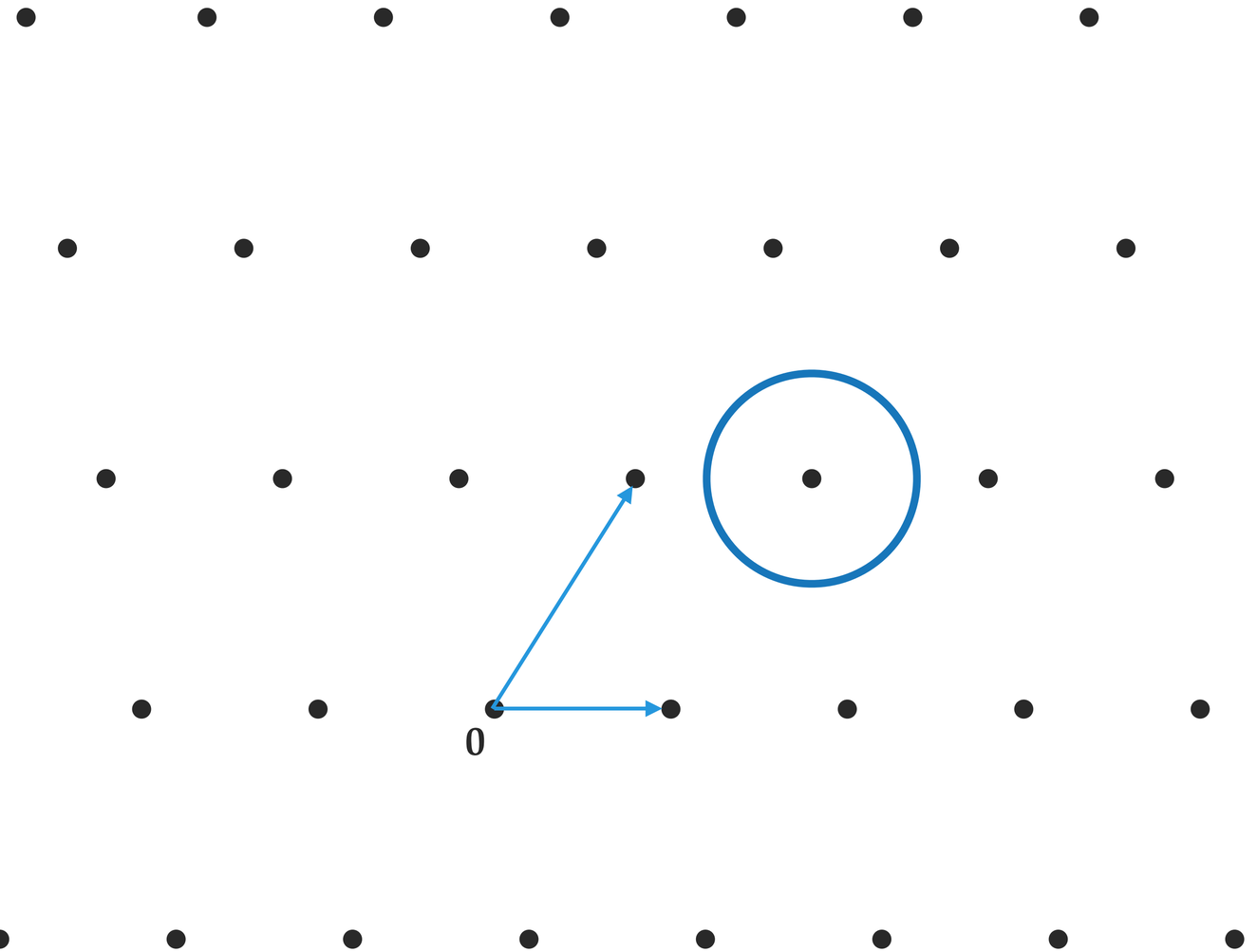
Pick any group of t columns of $\mathbf{H}$, add them and compare with $\mathbf{s}$.

Cost: $\binom{n}{t}$ sums of t columns.

Lattice-based cryptography



Lattice-based cryptography



A lattice $L \subset \mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$.

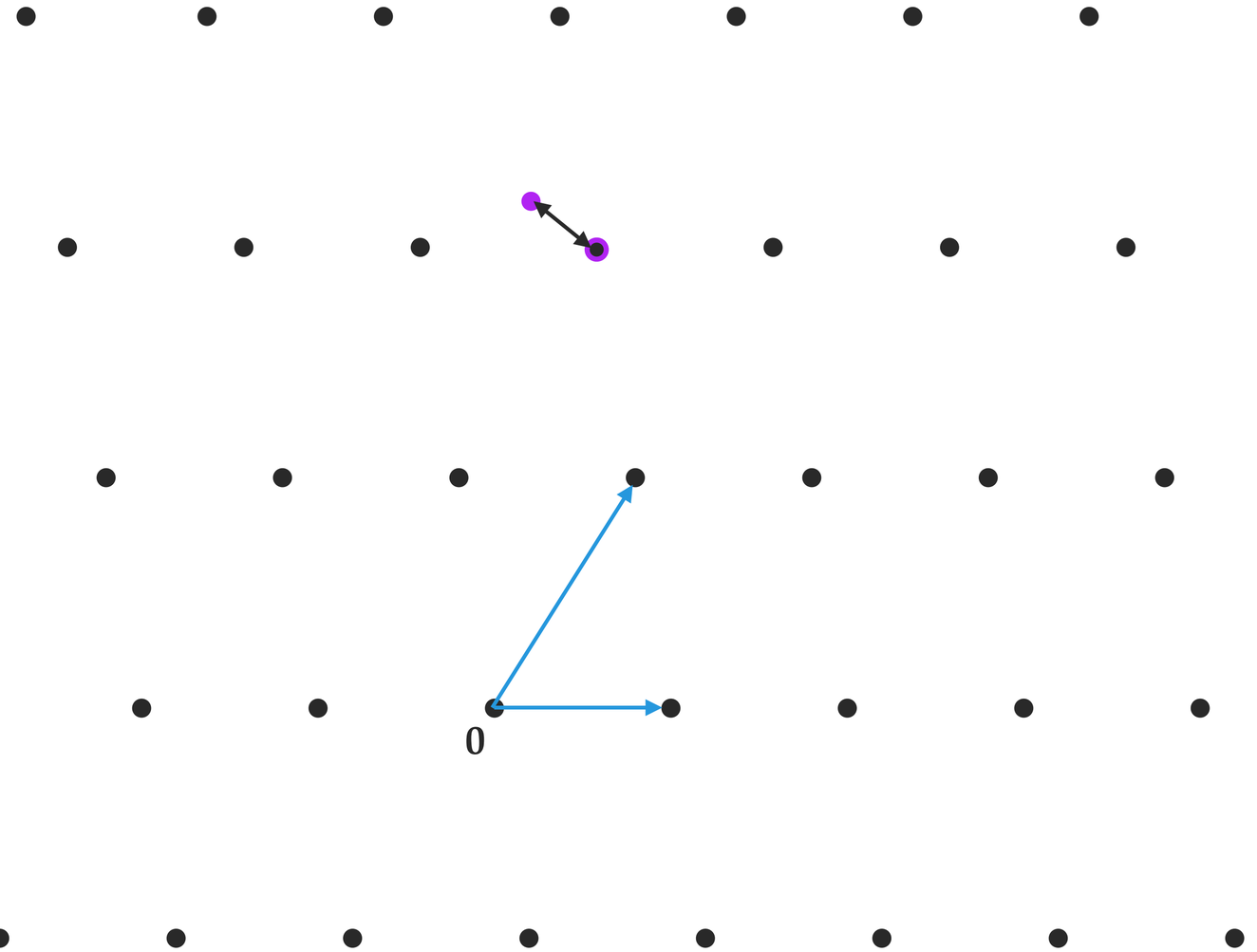
- dots: points on the lattice $\mathbf{c} \in L$.
- for every $\mathbf{v} \in L$, there exists an open ball around $\mathbf{v}$ that contains no other elements from L.

Lattice basis: n $\mathbb{R}$-linearly independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$

$$L := \{ \sum_{i=1}^{n} a_i \mathbf{b}_i \,|\, a_i \in \mathbb{Z} \}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & 5 \end{pmatrix}$$

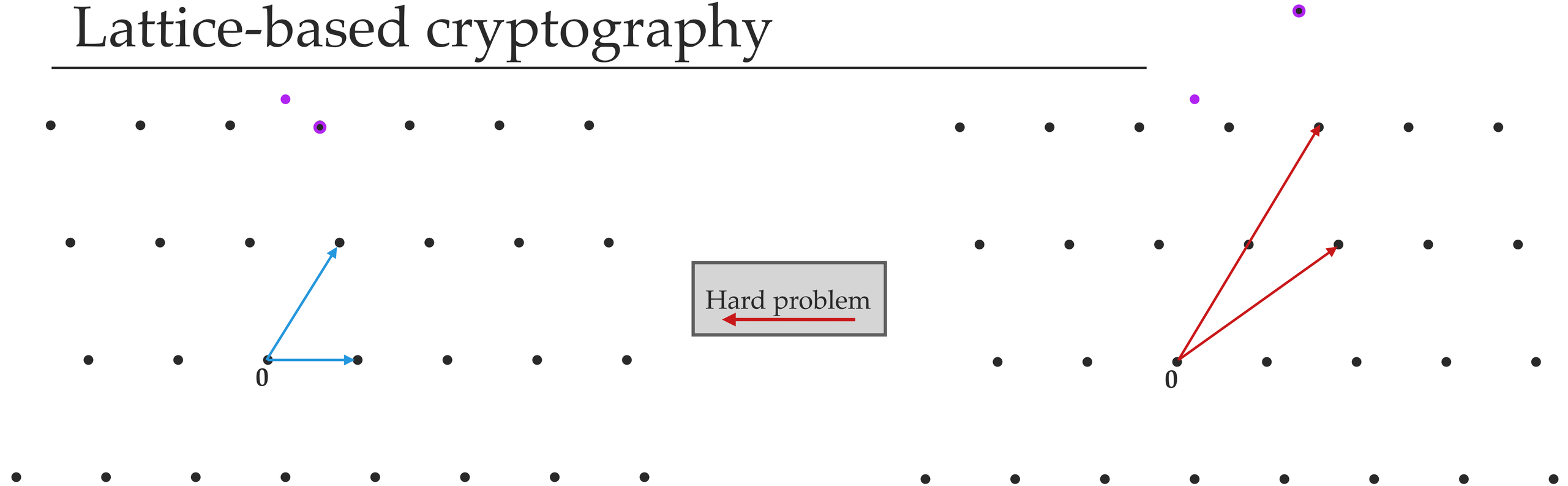
Lattice-based cryptography



The Closest Vector Problem (CVP)

Input: an arbitrary basis **B** of a lattice L and a target vector $\mathbf{t} \in \mathbb{R}^n$.
Question: Find a lattice vector $\mathbf{v} \in L$ that is closest to **t**.

Lattice-based cryptography



Good basis

$$(\lambda_1, \lambda_2)\begin{pmatrix} 4 & 0 \\ 3 & 5 \end{pmatrix} = (1,11)$$

$\mathbf{t} = -1.4\,\mathbf{b}_1 + 2.2\,\mathbf{b}_2$

↓ rounding

$\mathbf{c} = -1\mathbf{b}_1 + 2\mathbf{b}_2$ ✓

CVP input $\mathbf{t} = (1,11)$

Bad basis

$$(\lambda_1, \lambda_2)\begin{pmatrix} 7 & 5 \\ 6 & 10 \end{pmatrix} = (1,11)$$

$\mathbf{t} = -1.4\,\mathbf{b}'_1 + 1.8\,\mathbf{b}'_2$

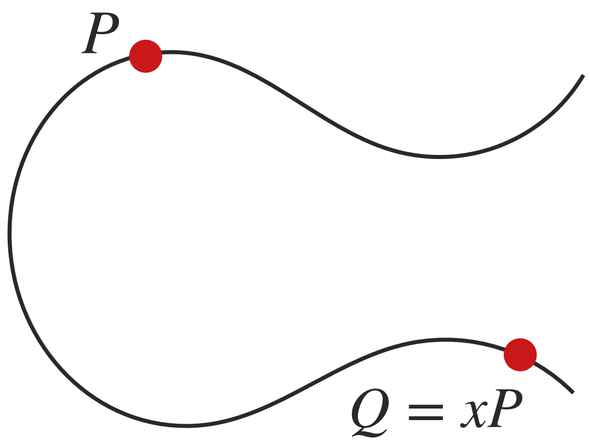
↓ rounding

$\mathbf{c} = -1\mathbf{b}'_1 + 2\mathbf{b}'_2$ ✗

Isogeny-based cryptography

Isogeny-based cryptography

- Elliptic curves

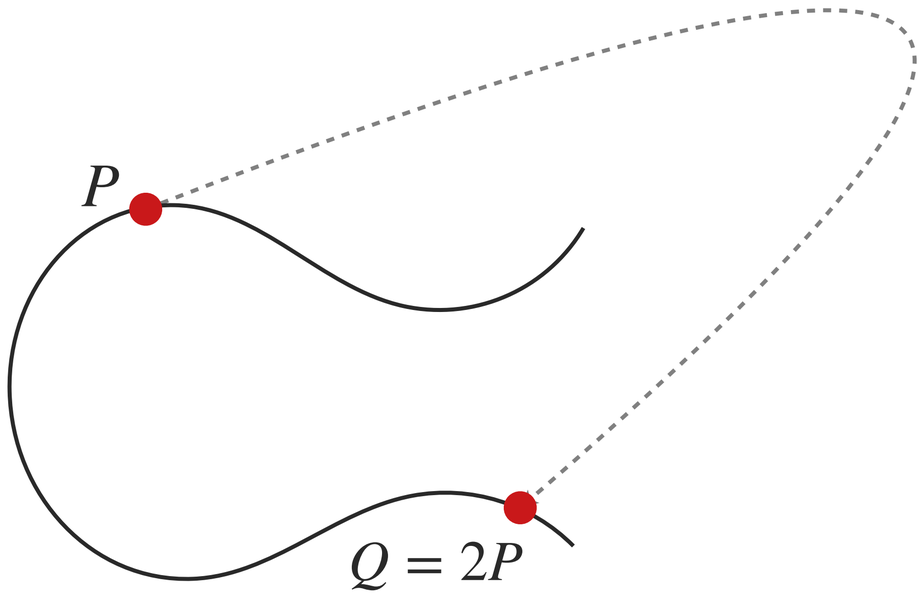


Isogeny-based cryptography

Elliptic curves

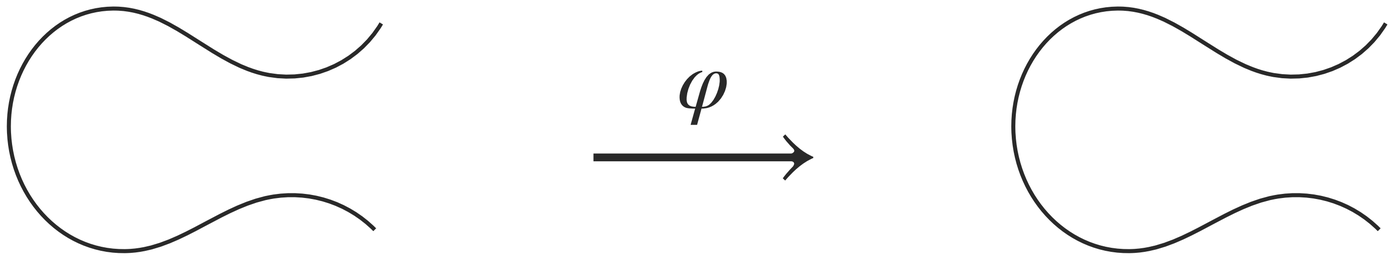
$$(x, y) \mapsto (\lambda^2 - 2x, \lambda x + y),$$

$$\lambda = \frac{3x^2 + a}{2y}$$



Isogeny-based cryptography

Isogenies: maps between elliptic curves



Isogeny-based cryptography

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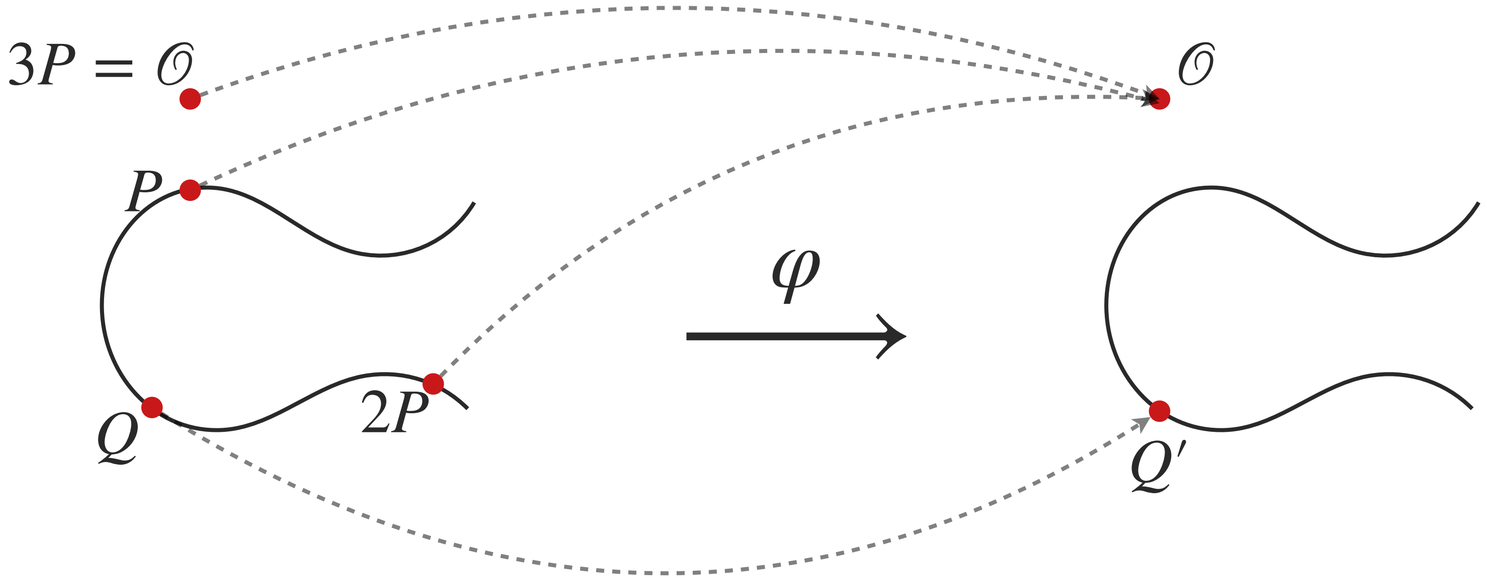


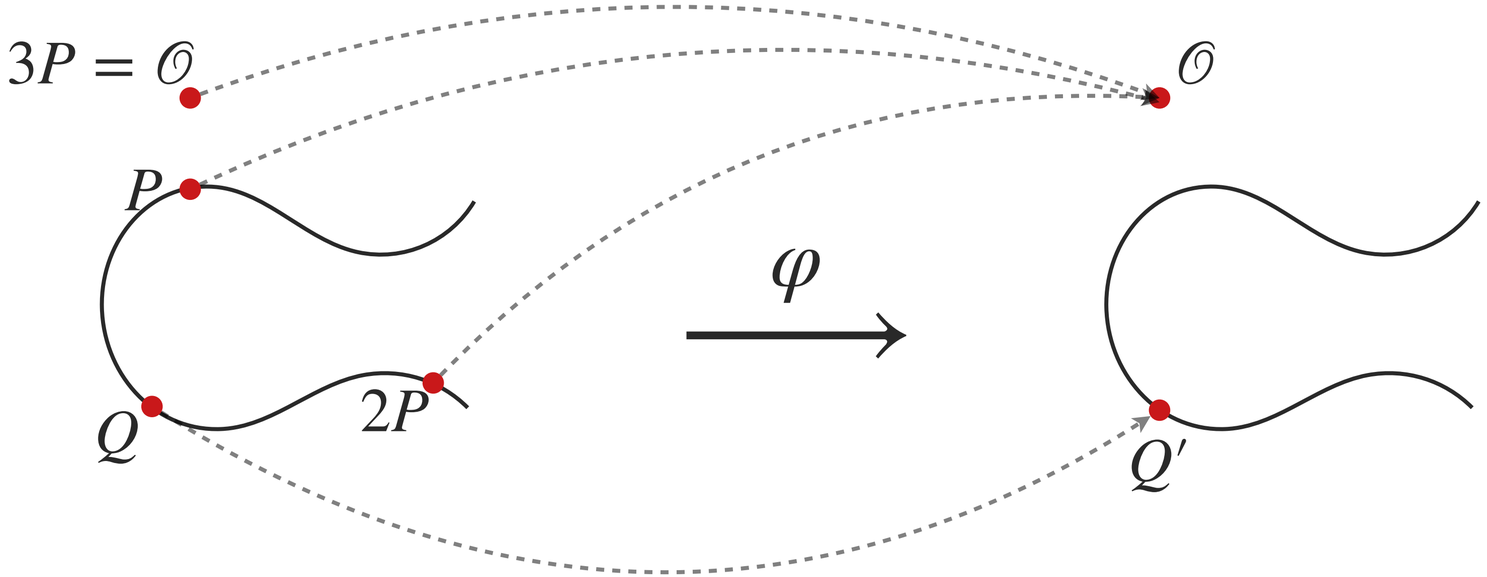
$$(x, y) \mapsto \left(\frac{x^3 - 4x^2 + 30x - 12}{(x-2)^2}, \frac{x^3 - 6x^2 - 14x + 35}{(x-2)^3} \cdot y\right)$$

Isogeny-based cryptography

Isogenies: maps between elliptic curves

The isogeny path problem

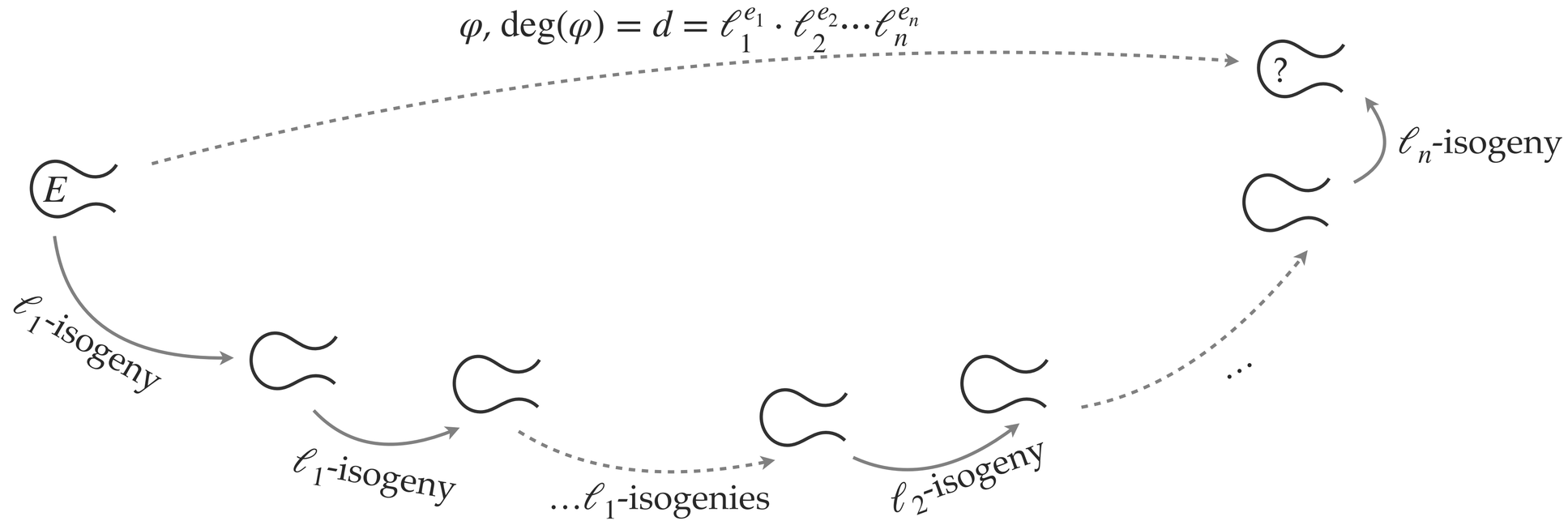
Input: Two supersingular curves E and E'.
Question: Find an isogeny φ from E to E'.



$$(x, y) \mapsto \left(\frac{x^3 - 4x^2 + 30x - 12}{(x-2)^2}, \frac{x^3 - 6x^2 - 14x + 35}{(x-2)^3} \cdot y\right)$$

Isogeny-based cryptography

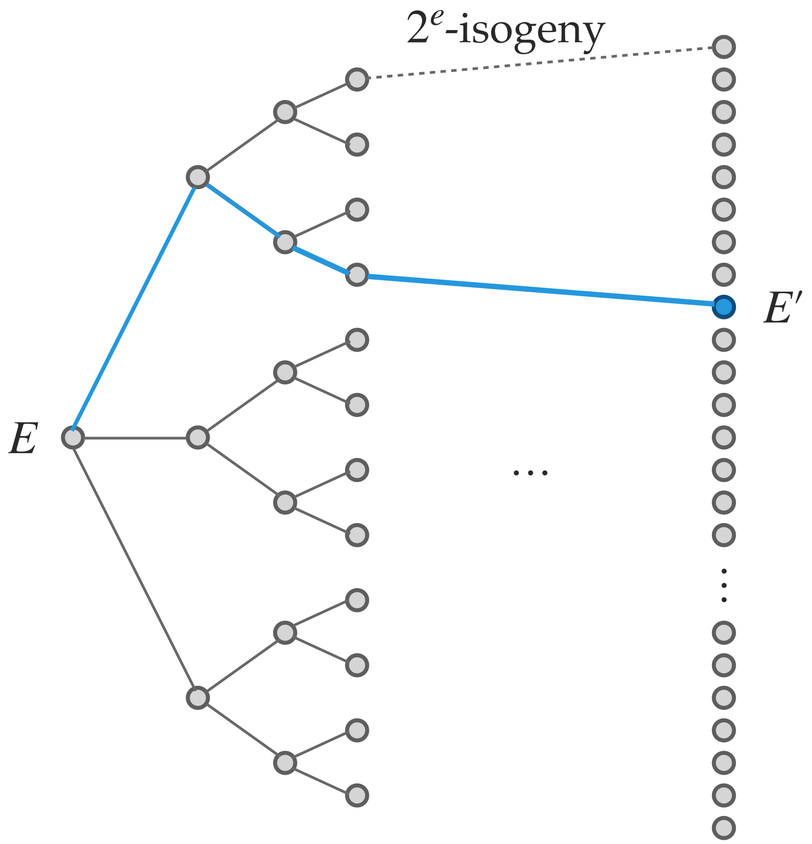
- Degree of an isogeny: how 'big' the isogeny is
  - Complexity of computing an isogeny: linear in the degree.
  - Composing isogenies: the degrees multiply: $\deg(\varphi \circ \psi) = \deg(\varphi) \cdot \deg(\psi)$.



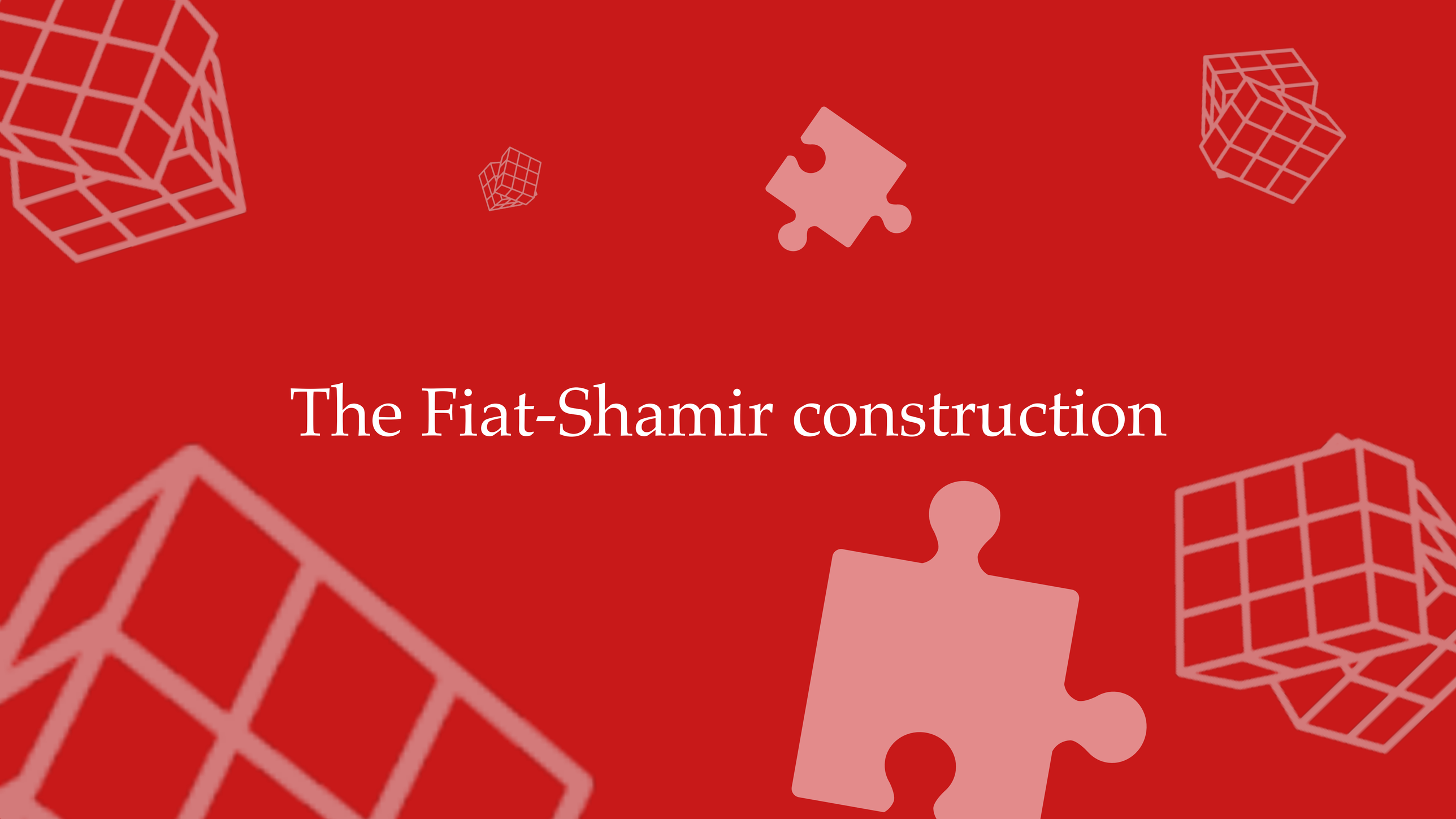
- From a curve E, there are $(\ell + 1)$ isogenies of degree ℓ.

Isogeny-based cryptography

Brute-forcing the (fixed-degree) isogeny path problem.



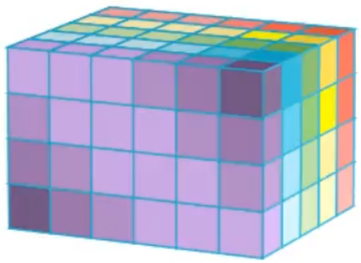
The Fiat-Shamir construction



Pick a hard problem

- 3-Tensor Isomorphism

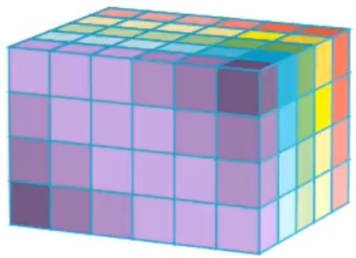
$$\mathcal{C} \subseteq \mathbb{F}_q^{m \times n \times k}$$



Pick a hard problem

3-Tensor Isomorphism

$\mathbf{T} \in \mathrm{GL}_k(q)$



$\mathbf{A} \in \mathrm{GL}_m(q)$

$\mathbf{B} \in \mathrm{GL}_n(q)$

MEDS

Pick a hard problem

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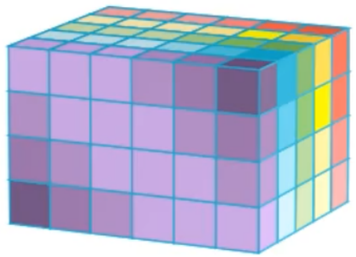

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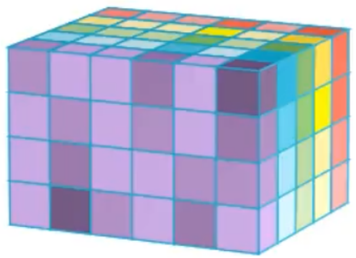

$\mathbf{A} \in \mathrm{GL}_m(q)$

$\mathbf{B} \in \mathrm{GL}_n(q)$

MEDS CD

Pick a hard problem

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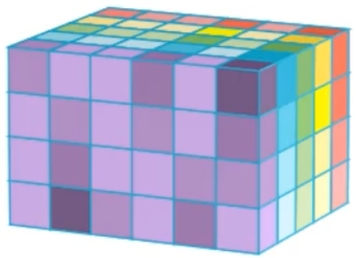

$\mathbf{A} \in \mathrm{GL}_m(q)$

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MEDS

Pick a hard problem

3-Tensor Isomorphism

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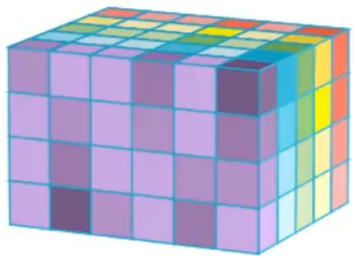

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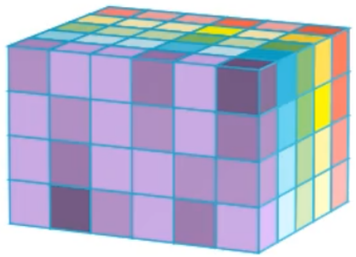


$\mathbf{A} \in \mathrm{GL}_m(q)$

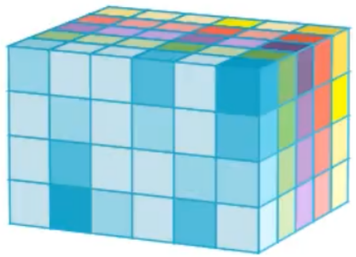
$\mathbf{B} \in \mathrm{GL}_n(q)$

MEDS

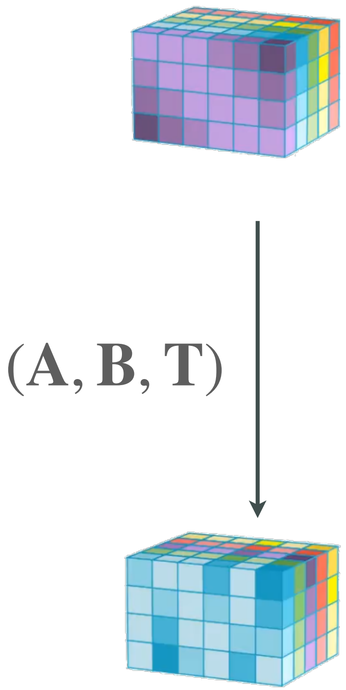
Pick a hard problem

- 3-Tensor Isomorphism

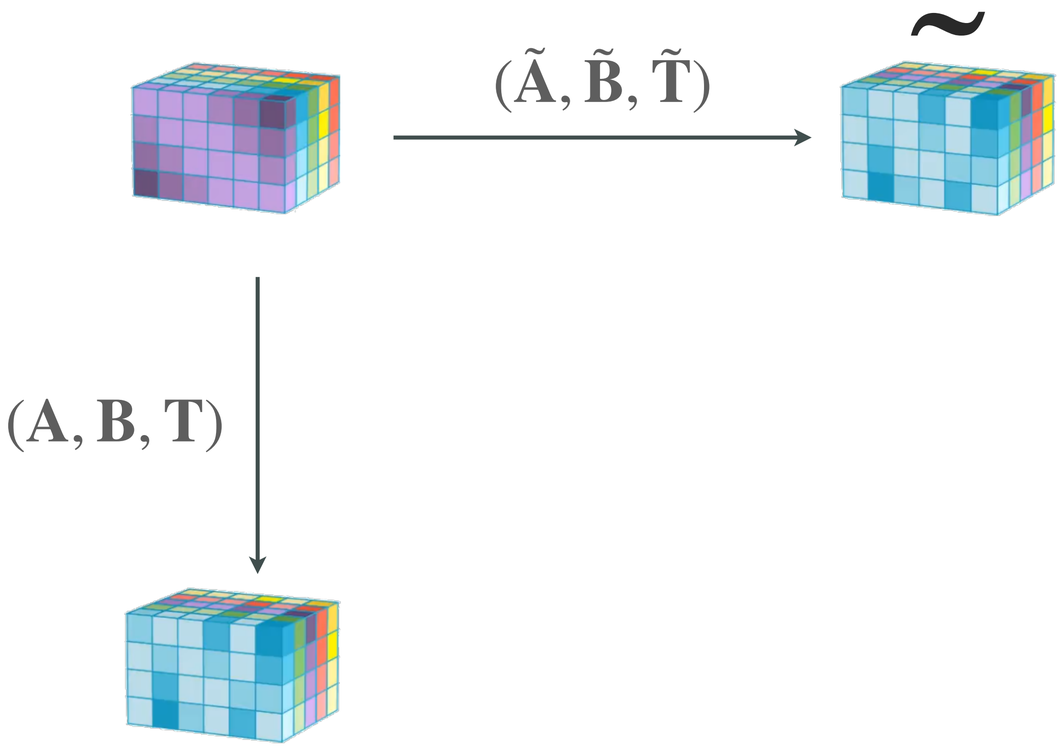
$$\mathcal{D} \subseteq \mathbb{F}_q^{m \times n \times k}$$



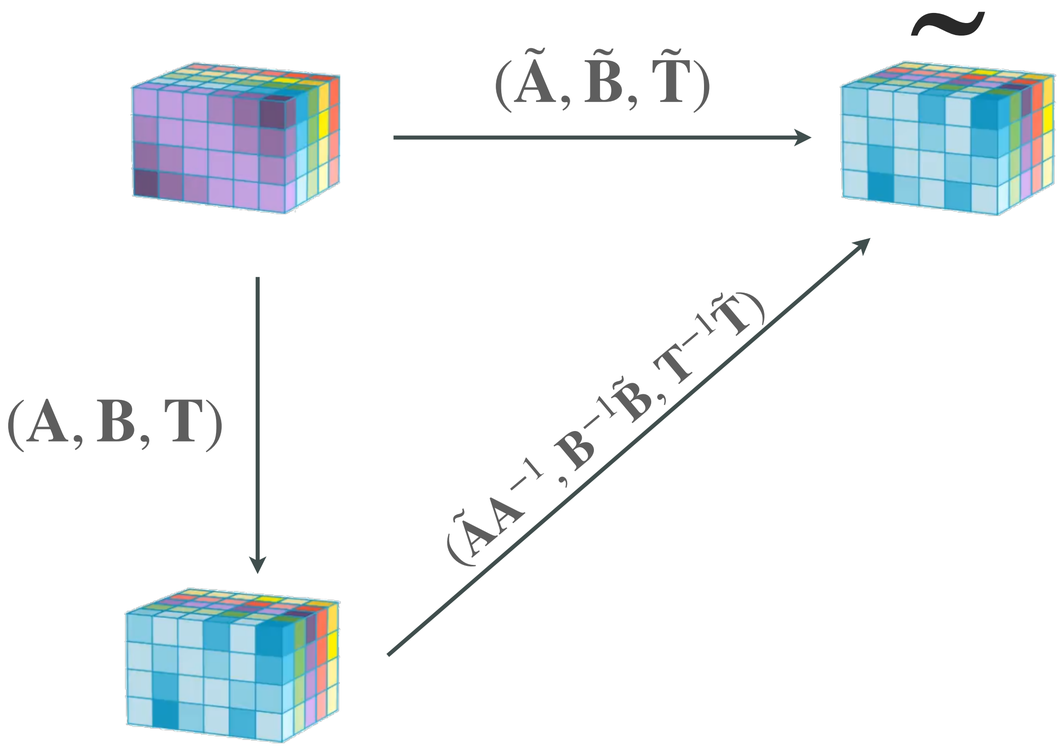
ZK identification scheme



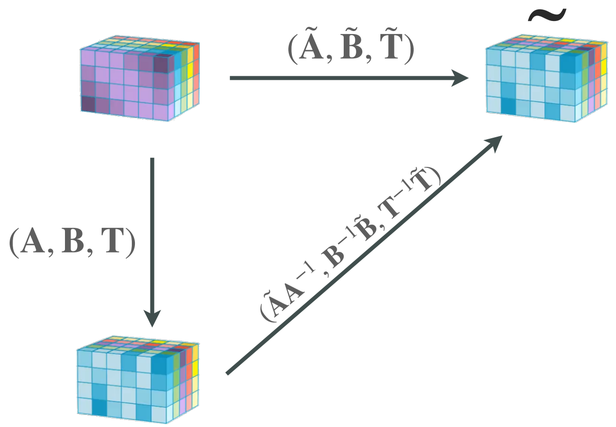
ZK identification scheme



ZK identification scheme



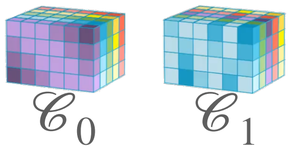
ZK identification scheme



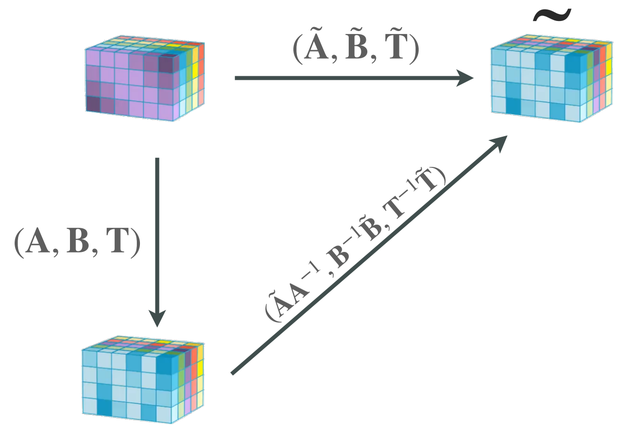
Prover

$(\mathbf{A}, \mathbf{B}, \mathbf{T})$

Verifier



ZK identification scheme

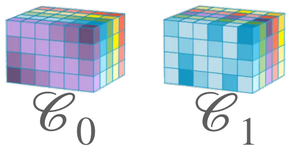


Prover

$(\mathbf{A}, \mathbf{B}, \mathbf{T})$

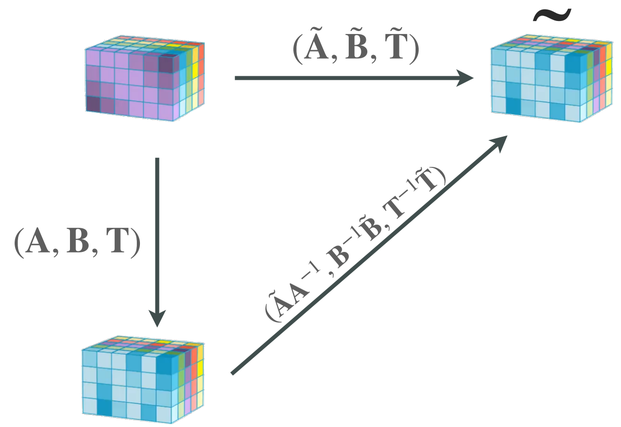
Commit to ephemeral $\tilde{}$

Verifier



MEDS

ZK identification scheme



Prover

Commit to ephemeral

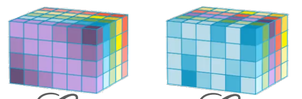
Verifier

Pick a challenge $b \in \{0,1\}$

A

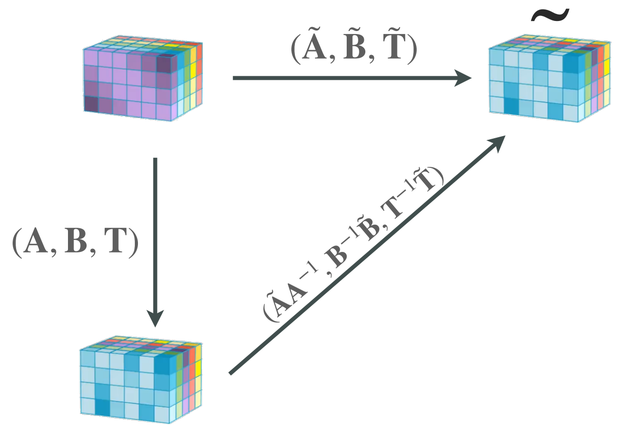
$(\mathbf{A}, \mathbf{B}, \mathbf{T})$

A



$\mathscr{C}_0$ $\mathscr{C}_1$

ZK identification scheme



Prover

Commit to ephemeral

Verifier

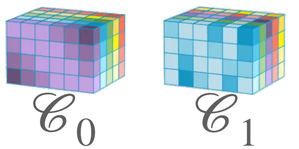
Pick a challenge $b \in \{0,1\}$

A

$(\mathbf{A}, \mathbf{B}, \mathbf{T})$

Response

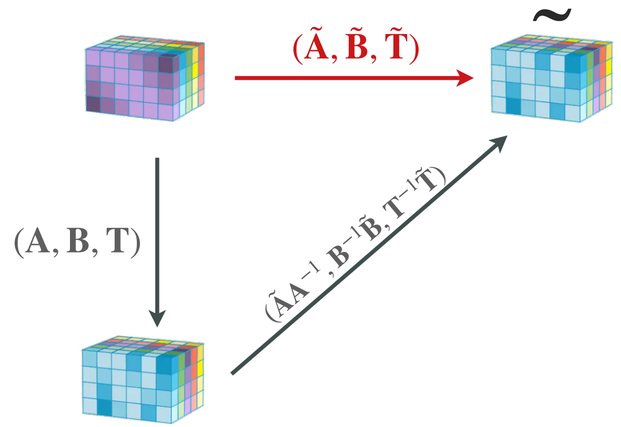
A



$\mathscr{C}_0$ $\mathscr{C}_1$

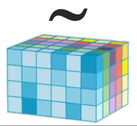
MEDS CD

ZK identification scheme



Prover

Commit to ephemeral



Verifier

Pick a challenge $b \in \{0,1\}$

$b = 0$

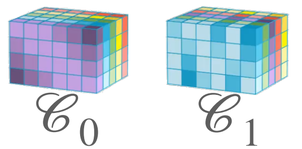
A

$(\mathbf{A}, \mathbf{B}, \mathbf{T})$

Response

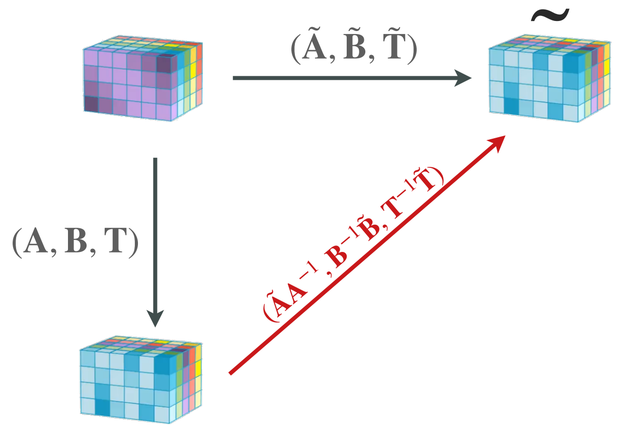
$(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{T}})$

A



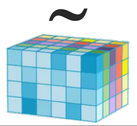
MEDS CD

ZK identification scheme



Prover

$(\mathbf{A}, \mathbf{B}, \mathbf{T})$

Commit to ephemeral 

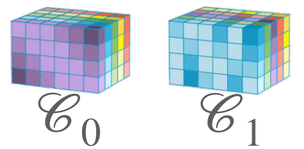
Pick a challenge $b \in \{0,1\}$

$b = 1$

Response

$(\tilde{\mathbf{A}}\mathbf{A}^{-1}, \mathbf{B}^{-1}\tilde{\mathbf{B}}, \mathbf{T}^{-1}\tilde{\mathbf{T}})$

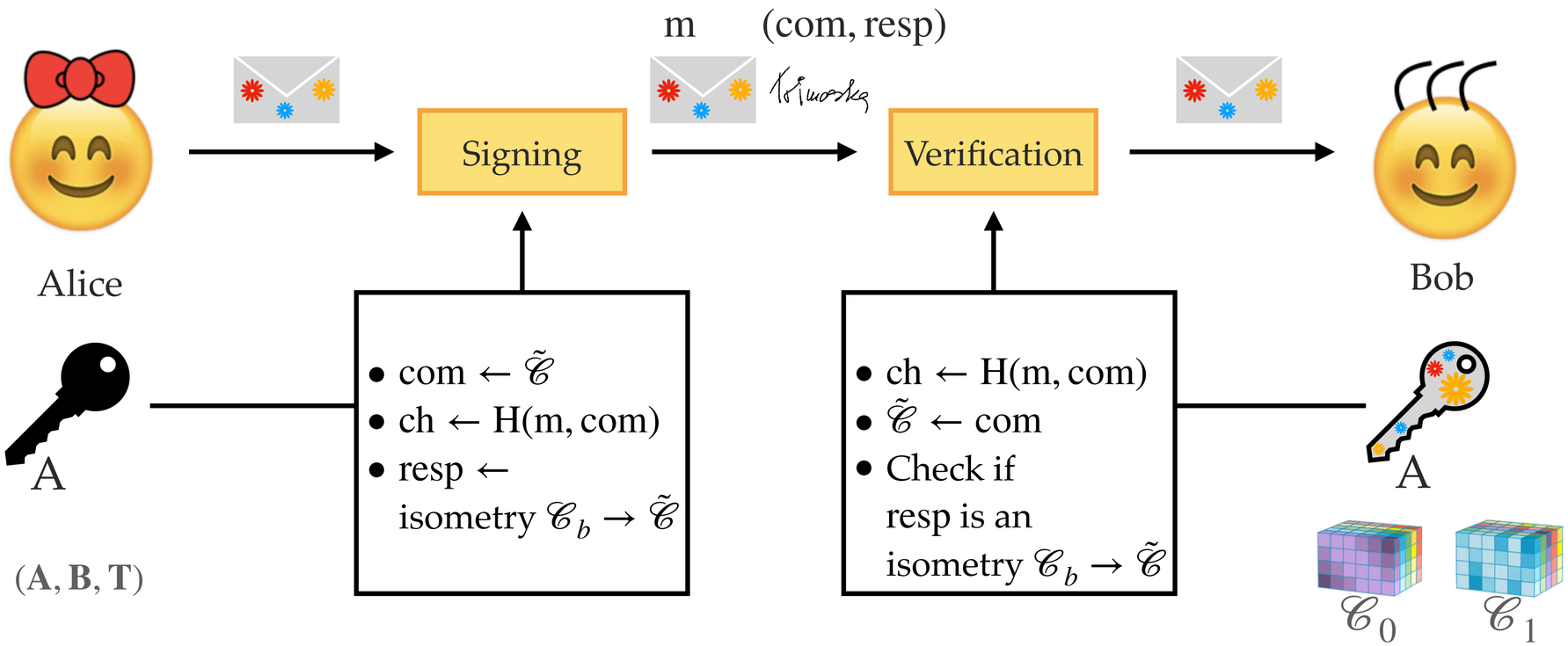
Verifier



The Fiat-Shamir transform

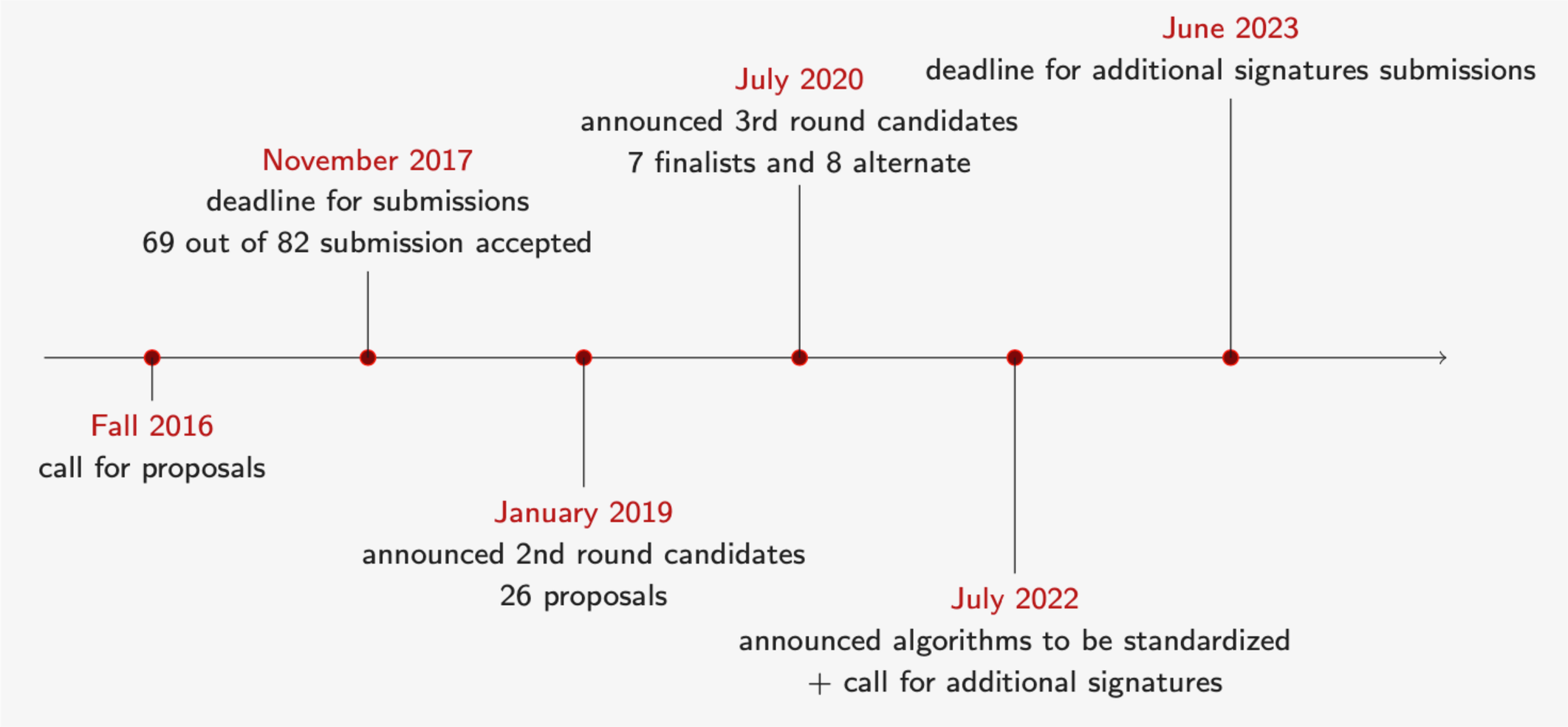
The goal is to transform an interactive identification scheme into a digital signature scheme.

Instead of the prover choosing a challenge, the challenge is determined by the hash of the message and commitments.

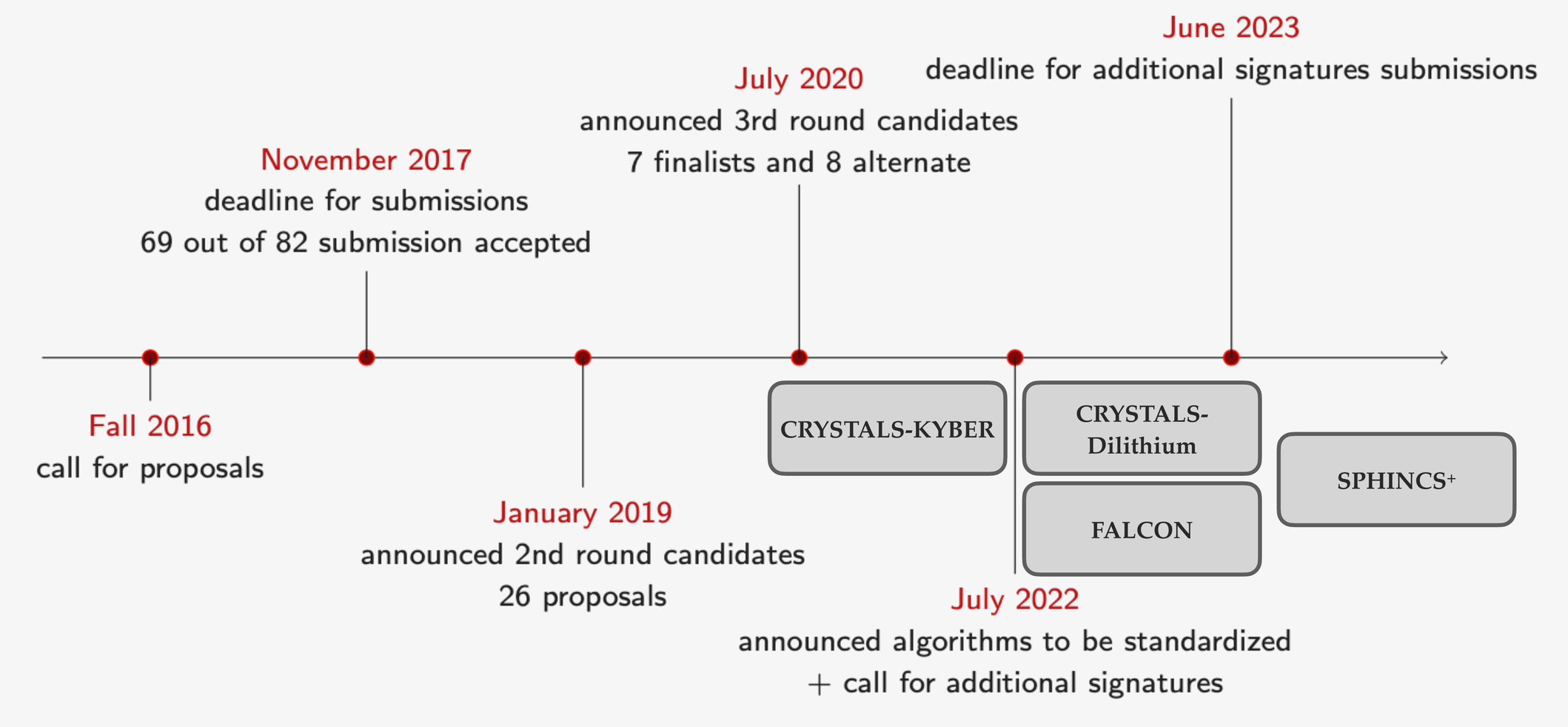


Timeline and challenges

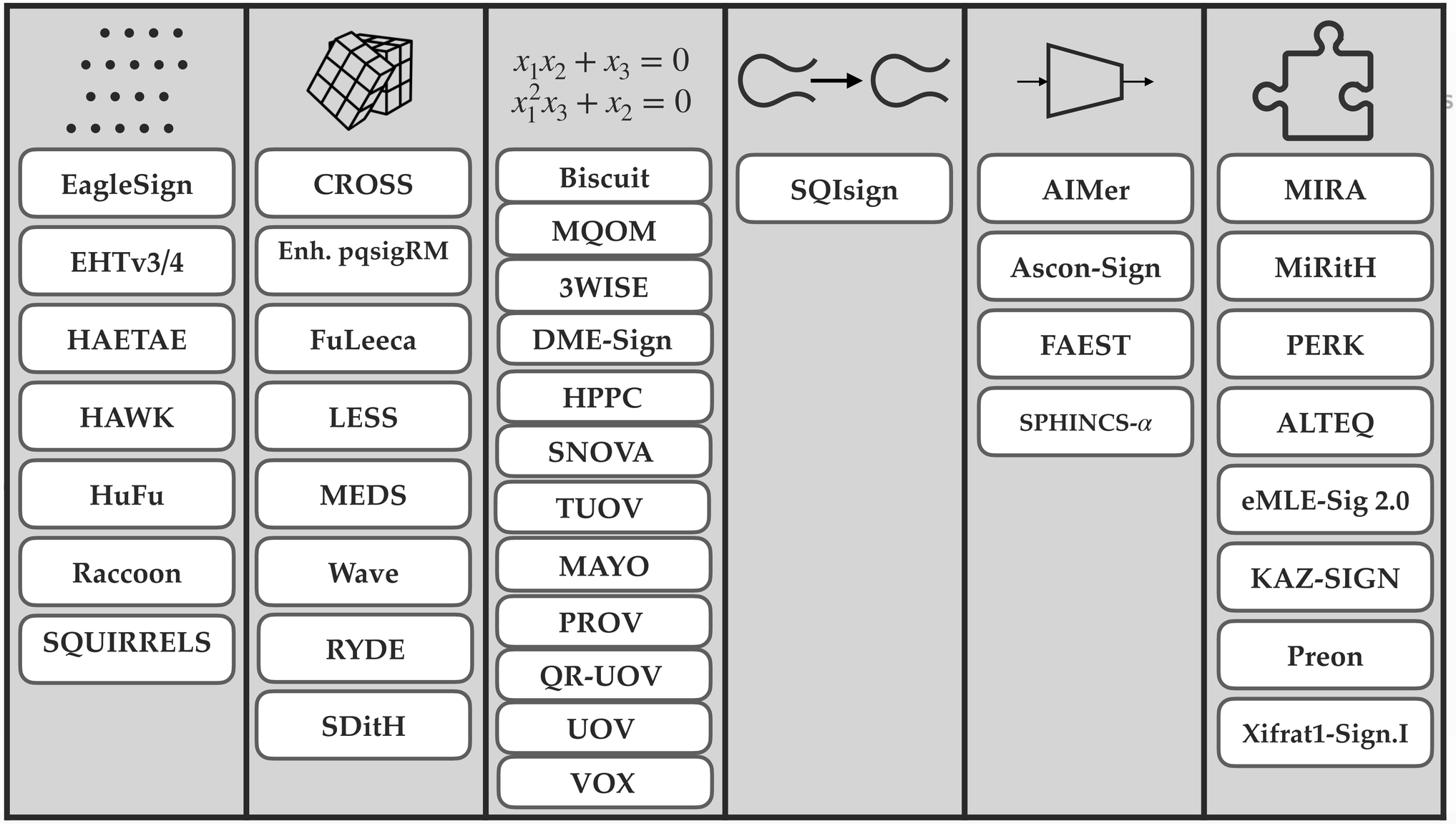
NIST standardisation timeline



NIST standardisation timeline

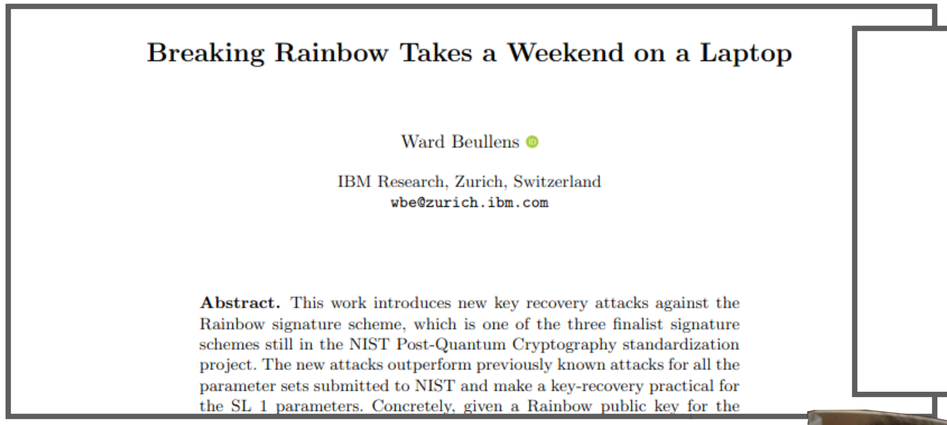


NIST standardisation timeline



Challenges in PQC

- Security assessment

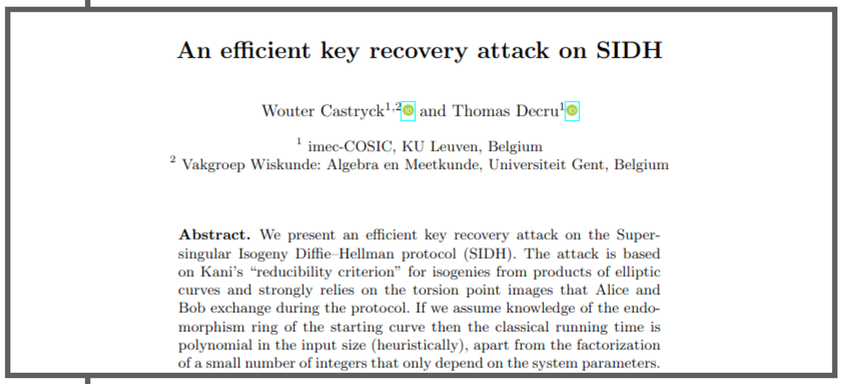


Breaking Rainbow Takes a Weekend on a Laptop

Ward Beullens

IBM Research, Zurich, Switzerland
wbe@zurich.ibm.com

Abstract. This work introduces new key recovery attacks against the Rainbow signature scheme, which is one of the three finalist signature schemes still in the NIST Post-Quantum Cryptography standardization project. The new attacks outperform previously known attacks for all the parameter sets submitted to NIST and make a key-recovery practical for the SL 1 parameters. Concretely, given a Rainbow public key for the



An efficient key recovery attack on SIDH

Wouter Castryck[1,2] and Thomas Decru[1]

[1] imec-COSIC, KU Leuven, Belgium
[2] Vakgroep Wiskunde: Algebra en Meetkunde, Universiteit Gent, Belgium

Abstract. We present an efficient key recovery attack on the Supersingular Isogeny Diffie–Hellman protocol (SIDH). The attack is based on Kani's "reducibility criterion" for isogenies from products of elliptic curves and strongly relies on the torsion point images that Alice and Bob exchange during the protocol. If we assume knowledge of the endomorphism ring of the starting curve then the classical running time is polynomial in the input size (heuristically), apart from the factorization of a small number of integers that only depend on the system parameters.

- Key/ciphertext/signature sizes and computational costs

- Physical security assessment

- Building advanced constructions

Enjoy the school!