Software implementation of ECC

Radboud University, Nijmegen, The Netherlands

June 4, 2015

Summer school on real-world crypto and privacy
Šibenik, Croatia

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The ECDLP

Definition

Given two points P and Q on an elliptic curve, such that $Q \in \langle P \rangle$, find an integer k such that $kP = Q$.

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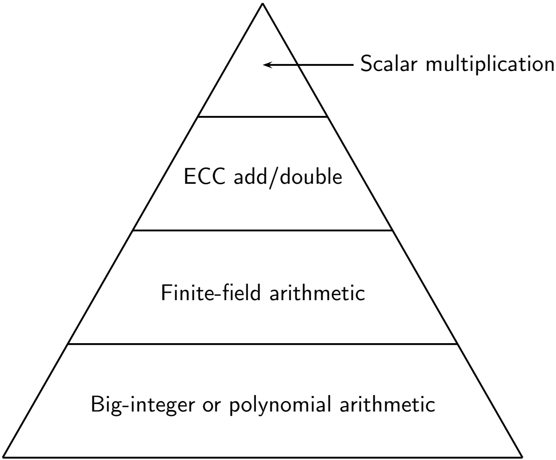
The ECC implementation pyramid



Why I don't like the pyramid...

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  - Correctness,
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  - Performance
- Plan for today: demonstrate these dependencies
- Fix target architecture: AMD64 (aka x86_64, aka x64)
- Fix target microarchitecture: Intel Sandy Bridge and Ivy Bridge

Let's start with 255-bit integers

```
typedef struct{
  unsigned long long a[4];
}  bigint255;

void bigint255_add(bigint255 *r,
                   const bigint255 *x,
                   const bigint255 *y)
{
  r->a[0] = x->a[0] + y->a[0];
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- What's wrong about this?
- This performs arithmetic on a vector of 4 independent 64-bit integers (modulo 2^{64})
- This is *not* the same as arithmetic on 256-bit integers
- Need to ripple the carries of all additions!

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- Example 2: When using vector arithmetic, carries are typically lost (expensive to recompute)
- Let's get rid of the carries, represent A as $(a_0, a_1, a_2, a_3, a_4)$ with

$$A = \sum_{i=0}^{4} a_i 2^{51 \cdot i}$$

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- This is called radix-2^{51} representation
- Multiple ways to write the same integer A, for example $A = 2^{52}$:
  - $(2^{52}, 0, 0, 0, 0)$
  - $(0, 2, 0, 0, 0)$

Addition of two bigint255

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- This works as long as all coefficients are in $[0, \ldots, 2^{63} - 1]$

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```

- This works as long as all coefficients are in $[0, \ldots, 2^{63} - 1]$
- When starting with 51-bit coefficients, we can do quite a few additions before we have to carry

Subtraction of two `bigint255`

```
typedef struct{
  signed long long a[5];
}  bigint255;

void bigint255_sub(bigint255 *r,
                   const bigint255 *x,
                   const bigint255 *y)
{
  r->a[0] = x->a[0] - y->a[0];
  r->a[1] = x->a[1] - y->a[1];
  r->a[2] = x->a[2] - y->a[2];
  r->a[3] = x->a[3] - y->a[3];
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}
```

- Slightly update our `bigint255` definition to work with *signed* 64-bit integers

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- Similar for all higher coefficients...

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- Carrying means evaluating at the radix
- Thinking of multiprecision integers as polynomials is very powerful for efficient arithmetic

Using floating-point limbs

- Now we can also use floats for our coefficients
- An IEEE-754 floating-point number has value

$$(-1)^s \cdot (1.b_{m-1}b_{m-2}\dots b_0) \cdot 2^{e-t} \text{ with } b_i \in \{0,1\}$$

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- For double-precision floats:
  - $s \in \{0,1\}$ "sign bit"
  - $m = 52$ "mantissa bits"
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- For single-precision floats:
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- Exponent $= 0$ used to represent 0
- Any number that can be represented like this, will be precise
- Other numbers will be *rounded*, according to a rounding mode

Addition

```
typedef struct{
  double a[12];
}  bigint255;

void bigint255_add(bigint255 *r,
                   const bigint255 *x,
                   const bigint255 *y)
{
  int i;
  for(i=0;i<12;i++)
    r->a[i] = x->a[i] + y->a[i];
}
```

Subtraction

```
typedef struct{
  double a[12];
} bigint255;

void bigint255_sub(bigint255 *r,
                   const bigint255 *x,
                   const bigint255 *y)
{
  int i;
  for(i=0;i<12;i++)
    r->a[i] = x->a[i] - y->a[i];
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```

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- Example: Radix 2^{22}, multiply by 2^{-22}
- This does *not* cut off lowest bits, need to round
- Some processors have efficient rounding instructions, e.g., `vroundpd`
- Otherwise (for double-precision):
  - add constant $2^{52} + 2^{51}$
  - subtract constant $2^{52} + 2^{51}$
  - This will round the number to an integer according to the rounding mode (to nearest, towards zero, away from zero, or truncate)

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- Operations on 256-bit vector registers introduced with AVX
- *Integer* operations on those registers introduced only with AVX2
- Sandy Bridge and Ivy Bridge don't have AVX2

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Computing multiple scalar multiplications

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Parallelism inside EC arithmetic

- Vectorize independent multiplications in EC addition
- May still need some shuffles (after each block of operations)
- Efficiency depends on EC formulas

Example: Montgomery ladder step

function ladderstep($x_{Q-P}, X_P, Z_P, X_Q, Z_Q$)

$t_1 \leftarrow X_P + Z_P$

$t_6 \leftarrow t_1^2$

$t_2 \leftarrow X_P - Z_P$

$t_7 \leftarrow t_2^2$

$t_5 \leftarrow t_6 - t_7$

$t_3 \leftarrow X_Q + Z_Q$

$t_4 \leftarrow X_Q - Z_Q$

$t_8 \leftarrow t_4 \cdot t_1$

$t_9 \leftarrow t_3 \cdot t_2$

$X_{P+Q} \leftarrow (t_8 + t_9)^2$

$Z_{P+Q} \leftarrow x_{Q-P} \cdot (t_8 - t_9)^2$

$X_{[2]P} \leftarrow t_6 \cdot t_7$

$Z_{[2]P} \leftarrow t_5 \cdot (t_7 + ((A+2)/4) \cdot t_5)$

return $(X_{[2]P}, Z_{[2]P}, X_{P+Q}, Z_{P+Q})$

end function

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$t_1 \leftarrow X_P + Z_P; t_2 \leftarrow X_P - Z_P; t_3 \leftarrow X_Q + Z_Q; t_4 \leftarrow X_Q - Z_Q$

$t_6 \leftarrow t_1 \cdot t_1; t_7 \leftarrow t_2 \cdot t_2; t_8 \leftarrow t_4 \cdot t_1; t_9 \leftarrow t_3 \cdot t_2$

$t_{10} \leftarrow ((A+2)/4) \cdot t_6$
$t_{11} \leftarrow ((A+2)/4 - 1) \cdot t_7$

$t_5 \leftarrow t_6 - t_7; t_4 \leftarrow t_{10} - t_{11}; t_1 \leftarrow t_8 - t_9; t_0 \leftarrow t_8 + t_9$

$Z_{[2]P} \leftarrow t_5 \cdot t_4; X_{P+Q} \leftarrow t_0^2; X_{[2]P} \leftarrow t_6 \cdot t_7; t_2 \leftarrow t_1 \cdot t_1$

$Z_{P+Q} \leftarrow x_{Q-P} \cdot t_2$

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A better candidate: Kummer surfaces

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- Easier way to think about it:
  - Group modulo negation
  - Map from group to Kummer surface by rational map X
  - Elements represented projectively as $(x : y : z : t)$
  - $(x : y : z : t) = (rx : ry : rz : rt)$ for any $r \neq 0$
  - Efficient doubling and efficient *differential addition*

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- Ladderstep: gets as input $X(P) = (x_2 : y_2 : z_2 : t_2)$, $X(Q) = (x_3 : y_3 : z_3 : t_3)$, and $X(Q - P) = (x_1 : y_1 : z_1 : t_1)$
  - Computes $X(2P) = (x_4 : y_4 : z_4 : t_4)$
  - Computes $X(P + Q) = (x_5 : y_5 : z_5 : t_5)$

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- Coordinates are elements of a (large) finite field

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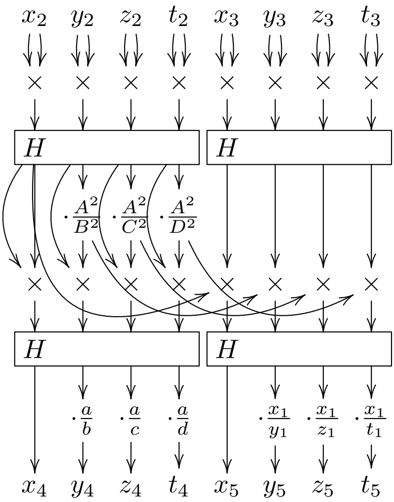
  - Computes $X(2P) = (x_4 : y_4 : z_4 : t_4)$
  - Computes $X(P + Q) = (x_5 : y_5 : z_5 : t_5)$
- Coordinates are elements of a (large) finite field
- For same security level, underlying field has half the size as for ECC
- Example: Choose ≈ 128-bit field for ≈ 128 bits of security

Arithmetic on the Kummer surface



$10\mathbf{M} + 9\mathbf{S} + 6\mathbf{m}$ ladder formulas

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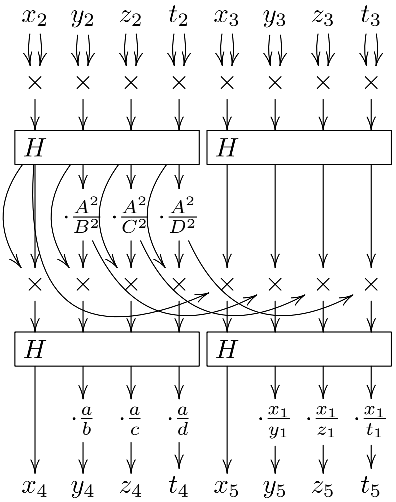

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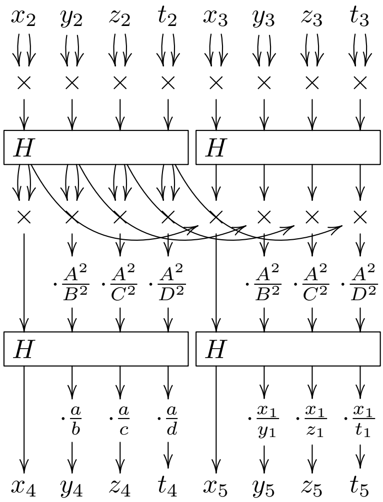
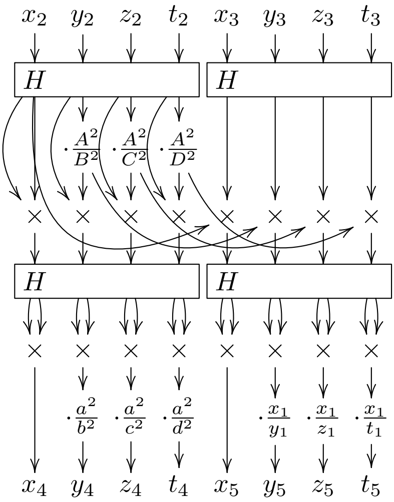


$7\mathbf{M} + 12\mathbf{S} + 9\mathbf{m}$ ladder formulas

The “squared Kummer surface”

- In fact, we use arithmetic on a different, “squared” surface
- Each point $(x : y : z : t)$ on the original surface corresponds to $(x^2 : y^2 : z^2 : t^2)$ on the squared surface
- No operation-count advantages
- Easier to construct squared surface with small constants
- In the following rename $(x^2 : y^2 : z^2 : t^2)$ to $(x : y : z : t)$

Arithmetic on the squared Kummer surface



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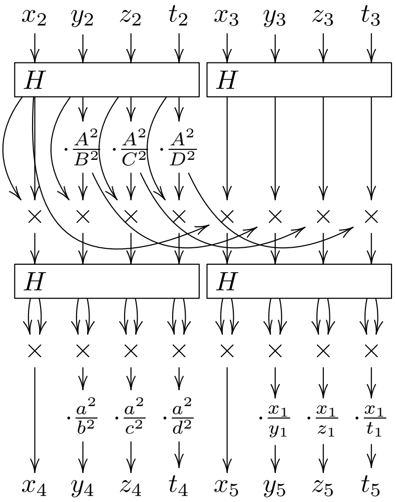

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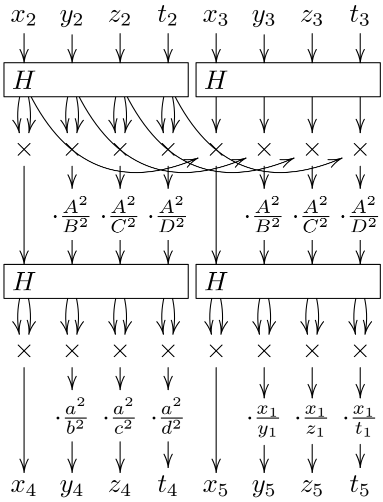
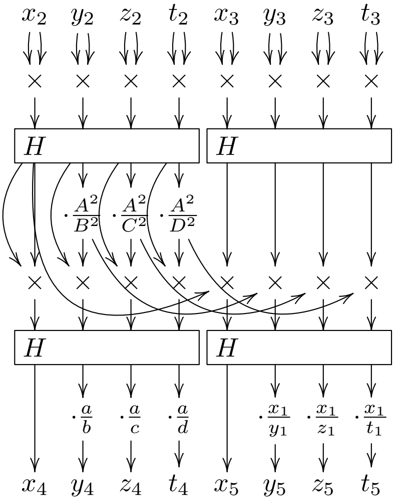
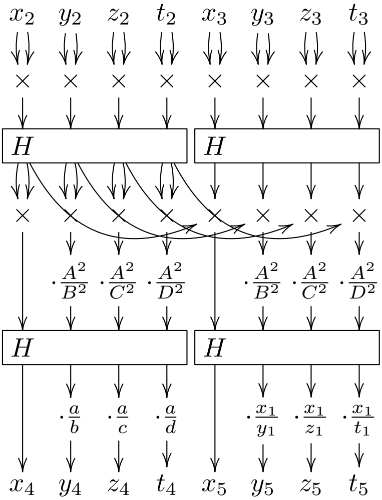


$7\mathbf{M} + 12\mathbf{S} + 9\mathbf{m}$ ladder formulas

Arithmetic on the (original) Kummer surface



$10\mathbf{M} + 9\mathbf{S} + 6\mathbf{m}$ ladder formulas



$7\mathbf{M} + 12\mathbf{S} + 9\mathbf{m}$ ladder formulas

A suitable Kummer surface

- Formulas for efficient Kummer surface arithmetic known for a while
  - Originally proposed by Chudnovsky, Chudnovsky, 1986
  - $10\mathbf{M} + 9\mathbf{S} + 6\mathbf{m}$ formulas by Gaudry, 2006
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  - Defined over the field $\mathbb{F}_{2^{127}-1}$
  - $(1 : a^2/b^2 : a^2/c^2 : a^2/d^2) = (-114 : 57 : 66 : 418)$
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- Finding this surface cost $1\,000\,000$ CPU hours
- The same surface has been used by Bos, Costello, Hisil, and Lauter (Eurocrypt 2013)

Representing elements of $\mathbb{F}_{2^{127}-1}$

- Represent an element A in radix-$2^{127/6}$
- Write A as $a_0, a_1, a_2, a_3, a_4, a_5$, where
  - a_0 is a small multiple of 2^0
  - a_1 is a small multiple of 2^{22}
  - a_2 is a small multiple of 2^{43}
  - a_3 is a small multiple of 2^{64}
  - a_4 is a small multiple of 2^{85}
  - a_5 is a small multiple of 2^{106}

Multiplication

- Consider multiplication of A and B with reduction mod $2^{127} - 1$
- Make use of the fact that $2^{127} \equiv 1$
- With radix $2^{127/6}$ we obtain:

$$
\begin{aligned}
r_0 &= a_0b_0 + 2^{-127}a_1b_5 + 2^{-127}a_2b_4 + 2^{-127}a_3b_3 + 2^{-127}a_4b_2 + 2^{-127}a_5b_1 \\
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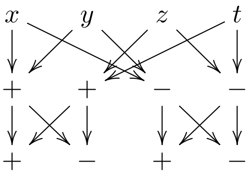
- Obviously, we always perform this whole thing $4\times$ in parallel
- Obviously, we specialize squaring
- Obviously, we specialize multiplications by small constants

The Hadamard transform



- Only shuffeling operation in Kummer arithmetic
- AVX has limited shuffeling across left and right half
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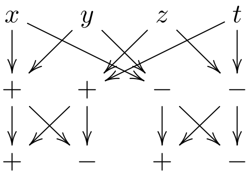

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- Self-inverting permutation "cleans" after two generalized Hadamards

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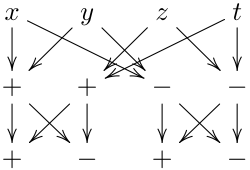

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Permuted and negated Hadamard

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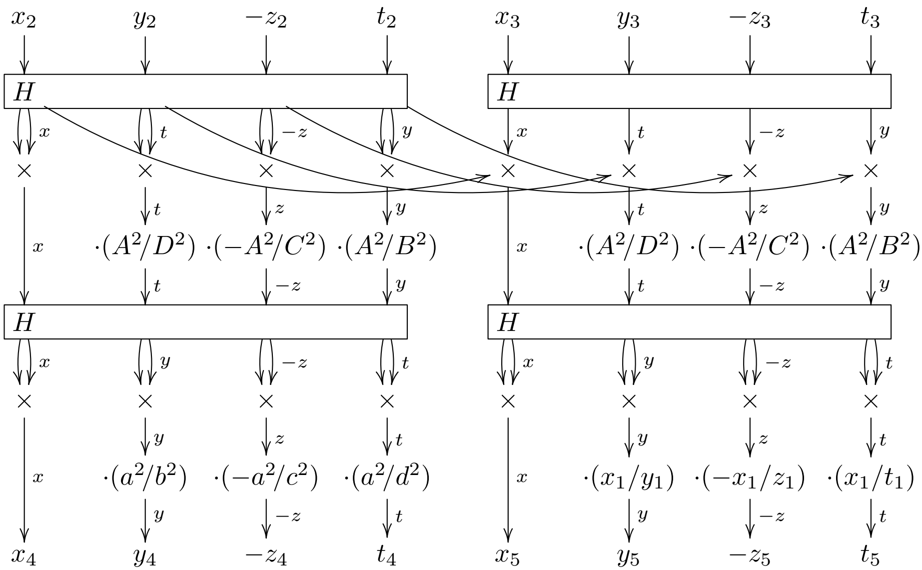
- Allow generalized Hadamard to negate vector entries
- "Clean" negations by multiplication by negated constants

Arithmetic on the squared Kummer surface



Looking back...

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- Optimizations go through all levels of the pyramid!

Results

128-bit secure, constant-time scalar multiplication

arch	cycles	open	g	source of software
Sandy	194036	yes	1	Bernstein–Duif–Lange–Schwabe–Yang CHES 2011
Sandy	153000?	no	1	Hamburg
Sandy	137000?	no	1	Longa–Sica Asiacrypt 2012
Sandy	122716	yes	2	Bos–Costello–Hisil–Lauter Eurocrypt 2013
Sandy	119904	yes	1	Oliveira–López–Aranha–Rodríguez-Henríquez CHES 2013
Sandy	96000?	no	1	Faz-Hernández–Longa–Sánchez CT-RSA 2014
Sandy	92000?	no	1	Faz-Hernández–Longa–Sánchez July 2014
Sandy	88916	yes	2	**new (our results)**

Results

128-bit secure, constant-time scalar multiplication

arch	cycles	open	g	source of software
Ivy	182708	yes	1	Bernstein–Duif–Lange–Schwabe–Yang CHES 2011
Ivy	145000?	yes	1	Costello–Hisil–Smith Eurocrypt 2014
Ivy	119032	yes	2	Bos–Costello–Hisil–Lauter Eurocrypt 2013
Ivy	114036	yes	1	Oliveira–López–Aranha–Rodríguez-Henríquez CHES 2013
Ivy	92000?	no	1	Faz-Hernández–Longa–Sánchez CT-RSA 2014
Ivy	89000?	no	1	Faz-Hernández–Longa–Sánchez July 2014
Ivy	88448	yes	2	**new (our results)**

More results

Also optimized for Intel Haswell

arch	cycles	open	g	source of software
Haswell	145907	yes	1	Bernstein–Duif–Lange–Schwabe–Yang CHES 2011
Haswell	100895	yes	2	Bos–Costello–Hisil–Lauter Eurocrypt 2013
Haswell	55595	no	1	Oliveira–López–Aranha–Rodríguez-Henríquez CHES 2013
Haswell	54389	yes	2	**new (our results)**

Even more results

Also optimized for ARM Cortex-A8

arch	cycles	open	g	source of software
A8-slow	497389	yes	1	Bernstein–Schwabe CHES 2012
A8-slow	305395	yes	2	**new (our result)**
A8-fast	460200	yes	1	Bernstein–Schwabe CHES 2012
A8-fast	273349	yes	2	**new (our result)**

Resources online

Paper:
Daniel J. Bernstein, Chitchanok Chuengsatiansup, Tanja Lange, Peter Schwabe. *"Kummer strikes back: new DH speed records"*.
http://cryptojedi.org/papers/#kummer

Software:
Included in SUPERCOP, subdirectory `crypto_scalarmult/kummer/`
http://bench.cr.yp.to/supercop.html